Contributions to the Theory of Groups

by John S. Rose

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ABSTRACT OF THESIS

Name of Candidate: John S. ROSE

Address: School of Mathematics, The University, Newcastle upon Tyne, NE1 7RU

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Title of Thesis: Contributions to the Theory of Groups

The thesis consists of eighteen papers on the theory of groups, together with an introduction which describes in outline the results obtained.

INTRODUCTION

A 1. The influence on a finite group of its proper abnormal structure, J. London Math. Soc. 40 (1965), 348-61; MR30#4838.

B 2. Abnormal depth and hypereccentric length in finite soluble groups, Math. Z. 90 (1965), 29-40; MR32#141.

C 3. On a splitting theorem of Gaschütz, Proc. Edinburgh Math. Soc. (2) 15 (1966), 57-60; MR33#5708.

A 4. Finite groups with prescribed Sylow tower subgroups, Proc. London Math. Soc. (3) 16 (1966), 577-89; MR33#5734.

B 5. Remarks on system normalizers and Carter subgroups, Proc. Internat. Conf. Theory of Groups (Canberra 1965) (1967), 303-5.

B 6. Finite soluble groups with pronormal system normalizers, Proc. London Math. Soc. (3) 17 (1967), 447-69; MR35#2967.

C 7. A natural setting for the extensions of a group with trivial centre by an arbitrary group, Enseignement Math. 13 (1967), 167-73; MR38#1179.

B 8. Nilpotent subgroups of finite soluble groups, Math. Z. 106 (1968), 97-112; MR40#5736.

B 9. Absolutely faithful group actions, Proc. Cambridge Philos. Soc. 66 (1969), 231-7; MR40#1465.

C10. On the splitting of extensions by a group of prime order, Math. Z. 117 (1970), 239-48; MR43#356.

C11. Splitting properties of group extensions, Proc. London Math. Soc. (3) 22 (1971), 1-23; MR43#7515.

C12. Extensions by a free abelian group of rank 2, Proc. Roy. Irish Acad. 71A (1971), 19-26. MR44#4097.

B13. A subnormal embedding theorem for finite groups, J. London Math. Soc. (2) 5 (1972) 253-9; MR47#326.

C14. Universal finite group extensions and a non-splitting theorem, Israel J. Math. 15 (1973) 375-83.

A15. Sufficient conditions for the existence of ordered Sylow towers in finite groups, J. Algebra 28 (1974) 116-26.

C16. Automorphism groups of groups with trivial centre, Proc. London Math. Soc.

C17. Frattini normal subgroups of finite groups, unpublished.

A18. On finite insoluble groups with nilpotent maximal subgroups, unpublished.

NOTE. Substantial parts of papers [1], [2] and [6] are included in my Ph.D. thesis (Cambridge, 1964); and [5] is a short summary of [6]. The other listed papers are separate from the Ph.D. thesis and from each other, and are my own independent work except where explicit references are made to other authors.

In this introduction the papers are discussed in three separate batches, labelled A, B, C in the list above. Those in batch A are concerned with sufficient conditions for the solubility of finite groups and the analysis of the soluble groups which satisfy these conditions, as well as of insoluble groups which almost satisfy them; those in batch B deal with the internal structure of finite soluble groups; and those in batch C with problems about group extensions.

The first paper [1] contains generalizations of a classical theorem of O.J. Schmidt (1924) and K. Iwassawa (1941) which asserts that a finite group G is soluble if all its proper subgroups are nilpotent. This result had already been generalized by B. Huppert (1954), who showed that the conclusion holds if the word 'nilpotent' in the hypothesis is replaced by the more general 'supersoluble'. In [1] the effect of restricting the structural condition on subgroups to a subset of the set of proper subgroups is investigated. It is shown that G is soluble if all its proper abnormal subgroups are nilpotent, but that the conclusion now fails if 'nilpotent' is replaced by 'supersoluble'. On the other hand, G is soluble if either all its proper self-normalizing subgroups are supersoluble or all its abnormal maximal subgroups are supersoluble and of prime-power index in G.

These investigations are continued in [4], where it is shown that a finite group G is soluble if all its proper abnormal subgroups are supersoluble and if the Sylow 2-subgroups of G are abelian. Further, the effect is considered of replacing 'supersoluble' in some of these results by the still more general 'Sylow tower groups of a specified type'.

Results of an arithmetical kind are proved about the structure of finite soluble groups all of whose proper abnormal subgroups are supersoluble but which are not themselves supersoluble.

It is a well known fact of finite group theory that a finite supersoluble group is a Sylow tower group of a special type called ordered. Huppert (1954) proved a partial generalization of this result in which he assumed that the groups in question had orders not divisible by 2 or 3. That some such condition is needed is shown by the alternating group A^4 of degree 4 and order 12, which does not have an ordered Sylow tower. In the paper [15] it is shown that one may relax the condition on the prime divisors of the orders of groups in Huppert's result and related results by assuming instead merely that the group A^4 is not involved in the groups in question. A deeper result in [15] is that a finite group G has an ordered Sylow tower if, for each prime divisor p of the order of G except perhaps the largest, G has no elementary abelian subgroup of order p^3; providing that neither A^4 nor B^{32} is involved in G, where B^{32} is a certain group defined in the paper and which has degree 32 and order 160. A corollary of the arguments used is that if P is a finite p-group which has no elementary abelian subgroup of order p^3 and if P has an automorphism of prime order $q \neq p$, then either q divides p^2-1 or $p = 2$ and $q = 5$.

Underlying the work in [1] and [4] are the important results of J.G. Thompson's thesis (1959), and in particular his theorem that a finite group is soluble if it possesses a nilpotent maximal subgroup of odd order. Examples are known of finite insoluble groups G which possess nilpotent maximal subgroups (necessarily of even order). Thompson established a classification theorem for such a group G in which the Sylow 2-subgroup T of a nilpotent maximal subgroup is either dihedral or generalized quaternion; and he conjectured that this condition on T was in fact redundant. Typically, in the known examples, the Sylow 2-subgroups of G are the nilpotent maximal subgroups. In [18] this observation is made precise: it is shown that a finite insoluble group G with a nilpotent maximal subgroup is, modulo a certain central subgroup of G, the direct product of a nilpotent group and an insoluble group whose Sylow 2-subgroups are maximal subgroups; and in particular, that if G has trivial centre, then the Sylow 2-subgroups of G are maximal subgroups. An example, described at length in the paper, shows that the clause 'modulo a certain central subgroup of G' cannot be omitted in general; but a positive result shows that the clause can validly be omitted under suitable conditions including those of Thompson's classification theorem. Another result yields a class of counter-examples to Thompson's conjecture.

In the first paper [2] of batch B, there is associated to every finite group G and every subgroup H of G a non-negative integer $a(G : H)$, called the abnormal depth of H in G. This integer is designed to give a measure of the complexity of the embedding of H in G. It has the properties (i) that $a(G:H) = 0$ if and only if H is a subnormal subgroup of G, (ii) that $a(G:H) \leq 1$ whenever H is a subgroup of G of prime power order, (iii) that if K is a subgroup of H then $a(G:K) \leq a(G:H) + a(H:K)$. The paper gives bounds for $a(G:H)$ under various conditions, with G soluble. In particular, it is shown that $a(G:H) \leq 1$ when G is metanilpotent and H is nilpotent. There is a contrast in the embedding properties of arbitrary subgroups between two different classes of metanilpotent groups. It is shown that if G is abelian-by-nilpotent then every subgroup H of G satisfies $a(G:H) \leq 1$; but that for every positive integer n, there is a supersoluble group G with a subgroup H such that $a(G:H) = n$. The proof of the latter fact uses a general characterization of the lattice of subgroups of the direct square $G \times G$ of an arbitrary group G which contain the diagonal subgroup of $G \times G$. This result has been included (and credited to the present author) by Huppert in his standard treatise ([d]†, p.52, Satz I.9.14).

Investigations of a similar kind are contained in [8], where there is a detailed study of the embedding of nilpotent subgroups in finite soluble groups of small nilpotent length. In order to show that various of the results obtained cannot be improved, examples are constructed, several of them by means of a technique which is treated axiomatically in [9]. There it is shown that if a finite group G acts on a finite abelian group A, where the orders of G and A are relatively prime, if the action is 'absolutely faithful' (as defined in the paper), and if K denotes the semi-direct product of

† References with small roman letters are listed at the end of the introduction.

A by G determined by this action, then G has exactly as many conjugacy classes of nilpotent subgroups as K has conjugacy classes of maximal nilpotent subgroups; and the correspondence between classes is described explicitly. It is also shown that this applies in particular if K is a regular wreath product of a non-trivial abelian group by a group G, where the orders of these groups are relatively prime, and A is the base group of K. This is the result needed in [8].

The note [5] is a brief summary of results in [6]. The paper [6] is concerned with rather more technical aspects of the theory of finite soluble groups. Every finite soluble group possesses two characteristic conjugacy classes of nilpotent subgroups which play a prominent rôle in the theory: the system normalizers (introduced by P. Hall in 1937) and the Carter subgroups (discovered by R.W. Carter in 1961). Every system normalizer is contained in a Carter subgroup and every Carter subgroup contains a system normalizer. The problem treated in [6] is to describe the relationship between system normalizers and Carter subgroups. This problem is still not satisfactorily resolved in general, but the results in [6] extend previous work of Carter. The paper introduces and exploits P. Hall's simple but important notion of a pronormal subgroup (which appears in the literature for the first time here). It also establishes a special property which has subsequently been made the basis of the definition of a now standard term (cf. G.A. Chambers [b]). This is a property possessed by the system normalizers of a soluble group G which has abelian Sylow p-subgroups for some prime p; namely, that the Sylow p-subgroups of the system normalizers are also Sylow p-subgroups of some normal subgroup of G. In the current terminology, one says that the system normalizers of G are 'p-normally embedded' in G. Some of the results of [6] have been generalized further by A. Mann (cf. [e] and [f]).

The final paper [13] in batch B deals with a different kind of question, which is not restricted to the class of soluble groups. The general problem, of which a special case is treated here, is the following. For a given group G, can G be embedded as a subnormal subgroup in some group K with prescribed properties? In [13], G is finite and K is to be finite and complete. It is shown that any finite group G can be embedded subnormally in a complete finite group

K; furthermore that certain properties of G can be imposed on K. In particular, if G is soluble then K can be made soluble. This shows that there is no bound on the complexity of complete finite soluble groups (for instance, on their orders, derived lengths, nilpotent lengths, etc.) A problem raised in [13], concerning the existence of non-trivial complete groups of odd orders, has recently been solved affirmatively by R.S. Dark ([c]).

Extension theory, to aspects of which the papers of batch C are devoted, aims at a description of the structure of all groups G containing a specified normal subgroup K with quotient group isomorphic to a specified group Q. Under favourable conditions, all such groups G 'split' over K, in which case there is a fairly satisfactory description of the groups G. An important problem of extension theory is therefore to identify sufficient conditions for splitting.

The paper [3] gives a new proof of a theorem of W. Gaschütz (1952) establishing particular sufficient conditions for splitting. The proof in [3] is more elementary than the original proof in that it relies on standard results and avoids factor system computations. [3] has been quoted in the book by A. Babakhanian ([a]).

Extensions of a group K by a group Q are usually classified into equivalence classes with respect to a standard equivalence relation. It is known that if K has trivial centre then the equivalence classes of extensions of K by an arbitrary group Q stand in one-to-one correspondence with the distinct homomorphisms of Q into the quotient of the group of all automorphisms of K by the group of inner automorphisms of K. In Kurosh's well known book, this fact is obtained as a corollary of deep cohomological results of S. Eilenberg and S. MacLane (1947). In [7] an entirely elementary proof is given, based on the observation that when K has trivial centre, copies of all the extensions of K by Q appear as subgroups of the direct product of Q and the full automorphism group of K. A simple consequence of the main result of [7] is the following theorem for finite groups. Let G be a non-trivial finite group and suppose that in a composition series of G there are just n_1 factors isomorphic to E_1, n_2 factors isomorphic to E_2, ..., n_k factors isomorphic to E_k, where E_1, ..., E_k are pairwise non-isomorphic simple groups, including all types of composition factors of G. Suppose further that every E_i is non-abelian and satisfies Schreier's Conjecture - this conjecture is correct for all known non-abelian finite simple groups - and also that every n_i is at most 4. Then G is isomorphic to the direct product of its composition factors (in a particular composition series).

It is convenient to consider next the paper [11]. This deals in a broad way with the problem of splitting of group extensions by seeking conditions on a given group K which ensure that all extensions of K by groups of some specified class split. Results obtained typically give necessary and sufficient conditions of the following kind: the centre of K must be restricted in some way and certain specified subgroups of the group of all automorphisms of K must split over the group of inner automorphisms of K. For instance, it is shown that all extensions of K split if and only if K has trivial centre and the full group of automorphisms of K splits over the group of inner automorphisms of K. Among the groups K satisfying these conditions one finds all complete groups; but there are also groups which are not complete. For example, it is proved in [11] that all extensions of the dihedral group of order 2n split (where n denotes an integer greater than 2) if and only if n is odd and at least one prime divisor of n is congruent to -1 modulo 4; whereas it is easy to show that the dihedral group of order 2n is complete if and only if n = 3. A detailed analysis of certain relative holomorphs of elementary abelian groups yields the following arithmetical result for finite groups. Suppose that a finite group G has an abelian minimal normal subgroup N, say of order p^n, and a chief factor L/N of prime order $q \neq p$ such that L is non-abelian. Let t be the least positive integer such that q divides p^t-1. If q^2 does not divide p^t-1 and q does not divide n then G splits over L. Examples show that these arithmetical conditions cannot be omitted.

It follows from results in [11] that if P is a finite p-group of exponent p^n, where p is a prime number and n a positive integer, there is an extension of P by a cyclic group of order p^n which does not split; moreover, if there is a central element of P which is not a p-th power of any element in P then there is an extension of P by a group of order p which does not split. One asks: do there exist non-trivial finite p-groups for which all extensions by a group of order p split? This question is treated in the paper [10] (commissioned for the Festschrift issue of Math. Z. for the sixtieth birthday of H. Wielandt). Necessary and sufficient conditions are established for all extensions of an arbitrary group K by a group of order p to split. This criterion is then applied to the case of a non-abelian p-group K with a cyclic subgroup of index p. It is shown that there is such a p-group K for which all extensions by a group of order p split providing that $p \neq 3$.

In [12] a rather more specialised problem is treated. It is known from work of R. Baer (1946) that for any group K with non-trivial centre there is an extension of K by a free abelian group of rank 2 such that K is not a direct factor of the extended group. In a similar vein, it is shown in [11] that if K is any group with non-trivial centre, there is an extension of K by an abelian group which does not split: the problem considered in [12] is whether, as in Baer's result, this abelian group can be taken to be free abelian of rank 2. Necessary and sufficient conditions are established for all extensions of an arbitrary group K by a free abelian group of rank 2 to split. Then this criterion is applied to demonstrate that all extensions of a dihedral group of order 8 by a free abelian group of rank 2 do in fact split; thus giving a negative answer to the question posed.

This is of interest because it shows (contrary to the expectations of several workers in the field) that in the Eilenberg-MacLane cohomological correspondence between the equivalence classes of extensions of a group K by a group Q with a specified coupling and the equivalence classes of extensions of the centre of K by Q with the induced coupling, split extensions of K by Q do not necessarily correspond to split extensions of the centre of K by Q.

In [14] the problem of splitting of group extensions is considered from a point of view dual to that of [11]: here one looks at the extensions of groups of some specified class by some given group G. Let G be a finite group. Then, according to the Schur-Zassenhaus theorem, a sufficient condition for all extensions by G of a finite group K to split is that the orders of G and K be relatively prime. However, this condition is in general not necessary, for it has been pointed out already that there are non-trivial finite groups all extensions of which split. Suppose now that G and K are finite groups whose orders have a common prime divisor. Then it is proved in [14] that there is a subgroup K* of a finite direct product of copies of K such that K is an epimorphic image of K* and such that there is an extension of K* by G which does not split. The structure of K* is quite similar to the structure of K: for instance, for every prime p, the Sylow p-subgroups of K and K* have the same class, derived length and exponent; and if K is soluble then K and K* have the same derived length, nilpotent length and p-length for all primes p. On the other hand, by the remark above, it is not in general possible to take K* = K; and in fact it is shown in [14] that it is not in general possible to find such a K* which is itself a direct product of copies of K. The proof of the main result in [14] depends on a wide generalization of a fundamental result of Gaschütz (1954): this establishes a universality property for suitable classes of group extensions.

It is clear that if one seeks to describe the extensions of a given group K, one needs adequate information about the automorphism group of K. The paper [16] is not directly concerned with group extensions, but with this problem, of basic importance for extension theory, of the determination of the full automorphism group of a given group. There is no systematic general procedure for doing this. The aim of [16] is to establish a reduction technique which provides a uniform method of calculating automorphism groups for various frequently occurring classes of groups with trivial centre. The key lemma gives the following statement. Let G be a group with a characteristic subgroup H whose centralizer in G is trivial. Then G can be identified in a natural way with a subgroup of the automorphism group of H, and this identification extends naturally to an identification of the automorphism group of G with the normalizer of G in the automorphism group of H. This is applied to the determination of automorphism groups of semi-simple groups, wreath products, and relative holomorphs of abelian groups. In particular, if A is a finite abelian group and G is a relative holomorph of A with trivial centre the results show that under various extra conditions the automorphism group of G is again a relative holomorph of A. An example shows that some extra condition is indispensable. An interesting corollary of these results is that the extended affine group of any finite field with more than 2 elements is a complete group. Moreover, for any positive integer m, there is a prime p such that in the extended affine group of the field with p^2 elements, which is itself complete, there is a chain of m distinct proper subgroups, all complete, and with prescribed odd prime indices.

Finally, [17] stems from Gaschütz's fundamental paper on extension theory in Crelle's Journal (1952). Gaschütz proved two parallel results by factor system calculations, one a criterion for splitting, the other for 'partial splitting' of a finite group over an abelian normal subgroup. Examples in the literature (the first due to Zassenhaus) shows that the splitting result cannot be generalized to give a similar criterion for splitting over a nilpotent normal subgroup. In [17] a new proof of the partial splitting result is given which shows that this criterion does remain valid for nilpotent normal subgroups. This leads to the question: if A is a normal p-subgroup of a finite group G contained in the Frattini subgroup of G, does A lie in the Frattini subgroup of a Sylow p-subgroup of G? It is shown that the answer is 'no' in general, but various positive results are proved under suitable conditions. An application is made to give information about dimensions of Gaschütz modules.

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A1

THE INFLUENCE ON A FINITE GROUP OF ITS PROPER ABNORMAL STRUCTURE

THE INFLUENCE ON A FINITE GROUP OF ITS PROPER ABNORMAL STRUCTURE

John S. Rose

1. *Introduction*

The results of the present paper spring from two sources: a classical theorem of O. J. Schmidt [**15**] and K. Iwasawa [**13**] and some later developments from it; and the theory of abnormal subgroups established by R. W. Carter in several papers—see in particular [**3**] and [**4**]. The Schmidt-Iwasawa theorem is the following:

I. *If a finite group G has all its proper subgroups nilpotent, then G is soluble.*

The hypothesis in (I) has in fact much stronger implications for the structure of G than solubility. For instance, Schmidt and Iwasawa showed that if in addition $|G|$ has at least 3 distinct prime factors, then G is actually nilpotent.

Among the many extensions of the Schmidt-Iwasawa theorem which have been published, one due to B. Huppert [**9**; Satz 22] is of particular interest. He showed that *nilpotent* in the statement (I) may be replaced by *supersoluble*, with the same conclusion; and that in this case, if $|G|$ has at least 4 distinct prime factors, then G is itself supersoluble. We shall consider the effects of replacing *proper* by *proper abnormal* in the hypotheses of these and similar results.

We show in Theorem 1 that (I) extends directly, but in Example 1 that Huppert's result does not: there exists an insoluble group, all of whose proper abnormal subgroups are supersoluble. This leads us to prove a weaker extension of Huppert's theorem, namely that if every proper self-normalizing subgroup of a finite group G is supersoluble, then G is soluble: this is the corollary to Theorem 3. The hypothesis that every proper abnormal subgroup of a finite group G is supersoluble is of course equivalent to the hypothesis that every abnormal maximal subgroup of G is supersoluble. We show finally, in Theorem 4, that if a finite group G has each of its abnormal maximal subgroups supersoluble and also of prime-power index in G, then G is soluble. This result may be viewed as a partial converse to the theorem of Galois which asserts that if G is a finite soluble group, then every maximal subgroup has prime-power index in G.

We make one remark on our methods of proof. There are two places where we use a deep result of J. G. Thompson [**18**], and a deduction from it, made in [**17**]. For convenience, these are stated here as:

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THEOREM A.† *Let G be a finite group and P a Sylow p-subgroup, where p is an odd prime. Let A be a group of automorphisms of G which leave P invariant. Suppose that for every A-invariant normal subgroup P_1 of P, elements of G of order prime to p which normalize P_1 actually centralize P_1. Then G possesses a normal p-complement.*

THEOREM B. *Let G be a finite group, and suppose that G has a maximal subgroup which is nilpotent and of odd order. Then G is soluble.*

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Definitions and notation. All groups considered will be finite. For a group G, we denote its order by $|G|$.

We write $H \leqslant G$ to mean H is a subgroup of G,

$H < G$ for H is a proper subgroup of G,

and $H \lhd G$ for H is a normal subgroup of G (not necessarily proper).

The index of a subgroup H in G is denoted by $|G : H|$.

$N_G(H)$ is the normalizer of H in G, and $C_G(H)$ the centralizer of H in G.

We define the *hypernormalizer* of H in G as follows:—

Let $H_0 = H$, and for each positive integer i, $H_i = N_G(H_{i-1})$. We have $H_0 \leqslant H_1 \leqslant H_2 \leqslant \dots$, and since G is finite, this ascending chain of subgroups terminates. The subgroup reached is the hypernormalizer of H. We denote this by $N_G^{\infty}(H)$.

If X is any non-void subset of G, $\{X\}$ denotes the subgroup of G generated by X.

H is an abnormal subgroup of G if, for each $g \in G$, we have $g \in \{H, g^{-1}Hg\}$; or equivalently, as shown by Carter in [**4**], if H satisfies the two conditions:

(a) every subgroup of G containing H is self-normalizing in G;

(b) H is not contained in two distinct conjugate subgroups of G.

We recall the obvious but convenient fact that a maximal subgroup of G is either normal or abnormal in G. Thus the abnormal maximal subgroups of G are precisely its non-normal maximal subgroups. Carter showed in [**4**] that a finite soluble group G always possesses nilpotent self-normalizing subgroups, that all such subgroups are conjugate and also abnormal in G. These are called the *Carter subgroups* of G.

† Thompson has published recently an improved result, with shorter proof, in *Journal of Algebra*, **1** (1964), 43–46. This yields a slightly shorter proof of our Theorem 2.

If ϖ is a set of prime numbers, we denote the complementary set by ϖ'. A ϖ-group is a group, every prime factor of whose order belongs to ϖ. When ϖ contains only one prime p, we refer to p-groups and p'-groups: a p-group is one of order a power of p, and a p'-group is one whose order is not divisible by p.

A subgroup H of a group G is a Hall ϖ-subgroup if it is a ϖ-group and $|G:H|$ is divisible by no prime in ϖ.

The Fitting subgroup of a group is its unique maximal nilpotent normal subgroup. The Frattini subgroup of a group is the intersection of all its maximal subgroups: this is well known to be a nilpotent characteristic subgroup, so that in particular it is contained in the Fitting subgroup.

2. *An extension of the Schmidt-Iwasawa theorem, and an example*

We begin with a lemma, which is a corollary of Thompson's Theorem B (above).

LEMMA 1. *If a finite insoluble group G has a nilpotent abnormal maximal subgroup M, then $|G|$ is even and $M = N_G(T)$ for some Sylow 2-subgroup T of G.*

Proof. It follows at once from Theorem B that $|G|$ is even. Let T be the Sylow 2-subgroup of M. $T \lhd M$. Either $M = N_G(T)$ or $T \lhd G$. If the latter were true, then G/T would have nilpotent maximal subgroup M/T of odd order, so that by Theorem B, G/T would be soluble and so G would be soluble, contrary to hypothesis. Therefore $M = N_G(T)$. If T were not a Sylow 2-subgroup of G, we should have $T < S$ for some Sylow 2-subgroup S of G, and $N_S(T) > T$; but $N_S(T)$ would be a 2-group contained in $N_G(T)$, while T is the Sylow 2-subgroup of $M = N_G(T)$—a contradiction. It follows that T is a Sylow 2-subgroup of G, and $M = N_G(T)$.

We may note that there exist insoluble groups having nilpotent abnormal maximal subgroups. For example, N. Itô **[12]** has pointed out that when p is a prime of the form $2^k - 1$ and $k \geqslant 4$, the projective special linear group $PSL(2, p)$ contains its Sylow 2-subgroups as maximal subgroups.

We have the following corollary to Lemma 1:

COROLLARY. *A finite group G can have at most one conjugacy class of nilpotent abnormal maximal subgroups.*

Proof. Let M_1, M_2 be nilpotent abnormal maximal subgroups of G. If G is soluble, then M_1, M_2 are Carter subgroups of G and therefore conjugate in G.

If G is insoluble, then, by Lemma 1, M_1, M_2 are the normalizers in G of conjugate subgroups of G; consequently M_1, M_2 are conjugate in G.

It is now easy to derive the following extension of the Schmidt-Iwasawa theorem.

THEOREM 1. *If every proper abnormal subgroup of the finite group G is nilpotent, then G is soluble. Moreover, G has a normal Sylow subgroup P such that G/P is nilpotent.*

Proof. We may assume G to be non-nilpotent. Then, by the Corollary to Lemma 1, G has a single conjugacy class of abnormal maximal subgroups. Let M be an abnormal maximal subgroup of G, and p a prime factor of $|G:M|$. Let P be a Sylow p-subgroup of G. Then $N_G(P)$ is abnormal in G, and p does not divide $|G:N_G(P)|$. So we must have $N_G(P)=G$, that is $P \lhd G$.

Now if L/P is a maximal subgroup of G/P, p does not divide $|G:L|$, and so $L \lhd G$. Every maximal subgroup of G/P is normal, so that by a well known characterization, G/P is nilpotent.

This result may also be proved without using Lemma 1, by induction on $|G|$ and application of the Schmidt-Iwasawa theorem to show that G cannot be simple, non-abelian. R. Baer [**1**; p. 124] has adopted essentially the latter method to prove a slightly different result, from which Theorem 1 follows easily.

We shall now show by an example that if we substitute *supersoluble* for *nilpotent*, the first assertion of Theorem 1 fails to hold.

EXAMPLE 1. *An insoluble group may have all its proper abnormal subgroups supersoluble.*

Consider the group $H=PGL(3, 2)$ of projectivities of a plane Π over the Galois field $GF(2)$. H is transitive as a permutation group on the seven points of Π, and also on the seven lines of Π. This accounts for the two conjugacy classes of octahedral subgroups of index 7 in H: they are the stabilizers of the points of Π and the stabilizers of the lines of Π. Note that $PGL(3, 2)=GL(3, 2)$, so that we may identify each projectivity with the unique matrix representing it, with respect to a fixed system of homogeneous coordinates for Π.

Then the stabilizer K of the point

$$\begin{bmatrix}1\\0\\0\end{bmatrix}$$

consists of all non-singular matrices

$$\begin{bmatrix}1 & a_{12} & a_{13}\\0 & a_{22} & a_{23}\\0 & a_{32} & a_{33}\end{bmatrix},$$

and the stabilizer $\bar{K}$ of the line [1 0 0] consists of all non-singular matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

The mapping $\alpha : x \to (x^{-1})'$ of H into itself (where, for any $y \in H$, y' is the transposed matrix of y) is clearly an automorphism of H of order 2. Moreover, α interchanges the subgroups K and $\bar{K}$ (so that α is an outer automorphism of H).

Form the subgroup $G = H\{\alpha\}$ of the holomorph of H, that is G is the split extension of H by α. Then the 14 octahedral subgroups of H form a single conjugacy class of self-normalizing subgroups of G. We assert that every maximal subgroup of G other than H is supersoluble.

Let M denote a maximal subgroup of G, $M \neq H$. Then $MH = G$, and so $|M : M \cap H| = 2$. Moreover, since H is a simple group (of order 168) and $1 < M \cap H < H$, we have $M = N_G(M \cap H)$. Therefore

$$N_H(M \cap H) = H \cap N_G(M \cap H) = M \cap H,$$

that is $M \cap H$ is self-normalizing in H. The subgroups of H are well known; see Burnside [**2**; Chap XX]. They fall into four sets:

(i) cyclic and dihedral subgroups;

(ii) subgroups of order 21—the normalizers of the Sylow 7-subgroups;

(iii) two classes of tetrahedral subgroups;

(iv) two classes of octahedral subgroups—the normalizers of the tetrahedral subgroups.

It is clear that an extension by a group of order 2 of any group of types (i) and (ii) is supersoluble. Also, since $M \cap H$ is self-normalizing in H, $M \cap H$ cannot be tetrahedral. So in order to verify the assertion, it only remains to show that $M \cap H$ cannot be octahedral. This follows from the fact that the octahedral subgroups of H are self-normalizing in G, whereas $M \cap H$ must have index 2 in its normalizer in G.

3. *Some sufficient conditions for p-solubility, and an extension of Huppert's theorem*

Before considering other possible extensions of Huppert's theorem, we shall prove some similar results for p-nilpotent and p-soluble groups. We recall that for a given prime p, a finite group G is called *p-nilpotent* if it possesses a normal p-complement; a finite group G is called *p-soluble* if there exists a series $1 = G_0 < G_1 < \ldots < G_r = G$ such that each $G_i \lhd G$ and each factor G_i/G_{i-1} is either a p-group or a p'-group. The *p-length*

$l_p(G)$ of a p-soluble group G is the least number of p-factors appearing in any series of G of the kind specified above.

The following analogue of the Schmidt-Iwasawa theorem is due to Itô [**11**], and is also proved by Huppert in [**10**];

II. *If a finite group G has all its proper subgroups p-nilpotent, then either G has normal Sylow p-subgroup or G is p-nilpotent. In particular, G is p-soluble and $l_p(G) = 1$ (if p divides $|G|$).*

We prove now a partial extension of this result.

THEOREM 2. *If every proper abnormal subgroup of the finite group G is p-nilpotent, and if in addition either* (i) *the Sylow p-subgroups of G are abelian, or* (ii) *p is an odd prime, then G is p-soluble. Furthermore, there exists in G a normal p-subgroup P_0 (possibly trivial) such that G/P_0 is p-nilpotent. If* (i) *is satisfied, G either has normal Sylow p-subgroup or is itself p-nilpotent. In any case, $l_p(G) \leqslant 2$.*

Proof. Let P be a Sylow p-subgroup of G. We may assume that $N_G(P) < G$, since otherwise the result is clear.

(i) Suppose P abelian. $N_G(P)$ is by hypothesis p-nilpotent, so that $N_G(P) = P \times R$ (direct product), where R is the normal p-complement of $N_G(P)$. Since P is abelian, P lies in the centre of $N_G(P)$. By a well known theorem of Burnside (see [**2**; p. 327]), this implies that G is p-nilpotent.

(ii) Now suppose p is an odd prime, and make no further assumption on the structure of P.

For any $x \in G$, write $\tau(x)$ for the inner automorphism of G induced by x. Let $A = \{\tau(x) : x \in N_G(P)\}$, a subgroup of the group of automorphisms of G. Then if P_1 is any A-invariant subgroup of P, we have $P_1 \lhd N_G(P)$, so that $N_G(P_1) \geqslant N_G(P)$. There are two possibilities:

(a) For every such $P_1 > 1$, $N_G(P_1) < G$. Then $N_G(P_1)$ is a proper abnormal subgroup of G, and so by hypothesis p-nilpotent. If R_1 is the normal p-complement of $N_G(P_1)$,

$$N_G(P_1) = P . R_1 \geqslant P_1 . R_1 = P_1 \times R_1.$$

R_1 centralizes P_1, so that $C_G(P_1) \geqslant R_1$. $|N_G(P_1) : C_G(P_1)|$ is therefore a divisor of $|N_G(P_1) : R_1| = |P|$, so that $N_G(P_1)/C_G(P_1)$ is a p-group. We may now apply Thompson's Theorem A to prove G p-nilpotent.

(b) Otherwise, for some (A-invariant) subgroup P_0 of P, $P_0 \neq 1$, we have $P_0 \lhd G$. Choose such a P_0 of maximal order. $P_0 < P$. Then G/P_0 is a group all of whose proper abnormal subgroups are p-nilpotent. P/P_0 is a Sylow p-subgroup of G/P_0 which is not normal in G/P_0. Moreover, if P_1/P_0 is a non-trivial subgroup of P/P_0, we cannot have $P_1/P_0 \lhd G/P_0$, for otherwise we should have $P_1 \lhd G$ with $P_0 < P_1 < P$, contradicting the prescribed maximality of P_0. By what we have proved already in (a), G/P_0 is p-nilpotent. This completes the proof of Theorem 2.

We can obtain a little more information about the structure of the group in Theorem 2. Consider any Sylow q-subgroup Q of G for any prime $q \neq p$ dividing $|G|$. Either $QP_0 \lhd G$; or $N_{G/P_0}(QP_0/P_0) < G/P_0$, in which case, since QP_0/P_0 is a Sylow subgroup of G/P_0, QP_0 lies in a proper abnormal subgroup of G, and is therefore by hypothesis p-nilpotent: consequently $QP_0 = Q \times P_0$ and Q centralizes P_0.

Let the prime factors of $|G|$ other than p be $q_1, \ldots, q_s$; and let Q_i be a Sylow q_i-subgroup of G $(i = 1, \ldots, s)$. We may suppose the primes labelled so that for $i = 1, \ldots, r$, $Q_i P_0 \lhd G$, while for $i = r+1, \ldots, s$, $Q_i P_0 \ntriangleleft G$, and Q_i centralizes P_0.

Write U/P_0 for the normal p-complement of G/P_0. Then U/P_0 has normal nilpotent Hall subgroup $H/P_0 = Q_1 P_0/P_0 \times \ldots \times Q_r P_0/P_0$. By Schur's theorem, U/P_0 splits over H/P_0, say $U/P_0 = H/P_0 . V/P_0$; and also V splits over P_0, say $V = P_0 . W$. W is a Hall subgroup of G which centralizes P_0, so that $V = P_0 \times W$.

COROLLARY. *Let U/P_0 be the normal p-complement of G/P_0. G/P_0 has a normal nilpotent Hall subgroup $H/P_0 \leqslant U/P_0$. U/P_0 splits over H/P_0, say $U/P_0 = H/P_0 . V/P_0$, and V has a direct product decomposition of the form $V = P_0 \times W$, where W is a Hall subgroup of G.*

We shall now show that we cannot improve the statement of Theorem 2 to conclude that $l_p(G) = 1$.

EXAMPLE 2. *For any odd prime p, there exists a group G satisfying the hypotheses of Theorem 2 and such that $l_p(G) = 2$.*

The construction of such a group is based on the result that if H is a finite group having a unique minimal normal subgroup K, and if q is any prime which does not divide $|K|$, then H has a faithful irreducible representation over any field F of characteristic q. In particular, if we take F to be the Galois field $GF(q)$, we may deduce that there exists an elementary abelian q-group Q, and a split extension H^* of Q by H, such that Q is the *unique* minimal normal subgroup of H^*. If we start with a cyclic group $H = H_1$, we can in this way form a sequence $H_1, H_2, \ldots$ of soluble groups with orders divisible by primes in a prescribed sequence, such that H_n has nilpotent length n, and every H_n has a unique chief series. These facts are described briefly by D. H. McLain in [**14**].

Given the prime p, let q be any prime $\neq p$. Then we can construct a group G with a unique chief series

$$G > H > K > 1$$

such that $|G/H| = p$, H/K is a q-group and K is a p-group. It is clear that $l_p(G) = 2$. Let M be an abnormal maximal subgroup of G. There are two possibilities:

(i) $M \geqslant K$. Then $MH = G$ and $M \cap H = K$ (since H/K is abelian and minimal normal in G/K). Therefore M is a Sylow p-subgroup of G.

(ii) $M \ngeqslant K$. Then $MK = G$ and $M \cap K = 1$ (since K is abelian and minimal normal in G). Then $M \cong G/K$, which is p-nilpotent.

Therefore every proper abnormal subgroup of G, being contained in an abnormal maximal subgroup of G, is p-nilpotent.

Example 1 shows that a group which is not 2-soluble may nevertheless have all its proper abnormal subgroups 2-nilpotent, so that Theorem 2 fails to hold without restriction on the Sylow 2-subgroups in the case $p = 2$. We prove next a weaker analogue of Theorem 2 which, however, provides new information in case $p = 2$.

THEOREM 3. *For any prime p, if every proper self-normalizing subgroup of a finite group G is p-nilpotent, then G is p-soluble. More precisely, there exists in G a normal p-subgroup P_0 such that G/P_0 is p-nilpotent;* $l_p(G) \leqslant 2$.

The success of this result when $p = 2$ is of course due to the fact that a proper self-normalizing subgroup need not be contained in an abnormal maximal subgroup of a group. In proving Theorem 3, we shall use a result which is stated and proved in a convenient form by G. Higman [**8**]:

LEMMA 2. *If the finite group G is not p-nilpotent, then it has a p-subgroup P, and an element x of order prime to p, such that $x \in N_G(P)$, but $x \notin C_G(P)$.*

We also need

LEMMA 3. *If a group G has a non-trivial subnormal p-subgroup P, then G has a non-trivial normal p-subgroup.*

Proof. Suppose $P = H_0 \lhd H_1 \lhd \ldots \lhd H_{r-1} \lhd H_r = G$, where r is a positive integer. We use induction on r. If $r = 1$, the assertion is trivial, so assume $r > 1$. By the induction hypothesis H_{r-1} has a non-trivial normal p-subgroup. Let P^* be the unique largest normal p-subgroup of H_{r-1}. $P^* \neq 1$. Then P^* is characteristic in H_{r-1}, and so $P^* \lhd G$. This completes the induction argument.

Proof of Theorem 3. We use induction on $|G|$. We may assume that G is not p-nilpotent, and then, since quotient groups of G also have the property that their proper self-normalizing subgroups are p-nilpotent, it is enough by the induction hypothesis to show that G has a non-trivial normal p-subgroup.

By Lemma 2, G has a p-subgroup P, and an element x of order prime to p, such that x normalizes but does not centralize P. Write $H = N_G{}^{\infty}(P)$, the hypernormalizer of P in G. By Lemma 3, if $H = G$ then G has a non-trivial normal p-subgroup. So suppose $H < G$. H is self-normalizing in G, so that by hypothesis H is p-nilpotent. Therefore $\{P, x\}$, as a subgroup of H, is p-nilpotent. But this implies that $\{P, x\} = P \times \{x\}$, which contradicts the fact that x does not centralize P. The result follows by induction.

The symmetric group of degree 4 is a group of 2-length 2, in which every proper self-normalizing subgroup is 2-nilpotent.

By making use of the celebrated result of W. Feit and J. G. Thompson announced in [**5**], we may state the result of Theorem 3 for the case $p = 2$ in the following way:

If every proper self-normalizing subgroup of a finite group G is 2-nilpotent, then G is soluble and has a factor H/K with $H \lhd G$, $K \lhd G$ and H/K isomorphic to a 2-complement of G.

The corollary of Theorem 3, stated below, is an immediate consequence. However, it is unnecessary to employ such a deep means of proof, and we prefer to derive the result by another method. We depend on the fact that a supersoluble group H is p-nilpotent, when p is the smallest prime factor of $|H|$: see, for instance, [**9**; Satz 7].

COROLLARY. *If every proper self-normalizing subgroup of a finite group G is supersoluble, then G is soluble.*

Proof. It is clear that quotients of G also have the property that their proper self-normalizing subgroups are supersoluble. We now observe that so also do subgroups of G. Let H be any subgroup of G, and V a proper self-normalizing subgroup of H. Then $N_G{}^\infty(V) < G$, for otherwise V would be subnormal in G, and so also in H, which is impossible. $N_G{}^\infty(V)$ is self-normalizing in G, and is therefore by hypothesis supersoluble. Hence V is supersoluble.

In proving the result by induction on $|G|$, it therefore suffices to show that G is not simple (we may suppose G non-cyclic). Let p be the smallest prime factor of $|G|$. Then every proper self-normalizing subgroup of G is p-nilpotent. By Theorem 3, G is p-soluble, and so G is not simple. The result follows by induction.

Remark. We may note that the argument in the first paragraph of the proof of this Corollary shows that if $\mathfrak{X}$ is any class of groups, closed under the operations of forming quotient-groups and subgroups, then the property that every proper self-normalizing subgroup of a finite group G belongs to $\mathfrak{X}$ is inherited by all the quotients and subgroups of G. Then an induction argument shows that if $\mathfrak{X}$ has the further property that a finite group G is soluble whenever every proper subgroup of G belongs to $\mathfrak{X}$, it is actually enough for G to be soluble that every proper *self-normalizing* subgroup belongs to $\mathfrak{X}$. The Corollary follows from Huppert's theorem by taking $\mathfrak{X}$ to be the class of supersoluble groups.

We have mentioned that Schmidt and Iwasawa proved that if every proper subgroup of G is nilpotent and $|G|$ has at least three distinct prime factors, then G is nilpotent; and Huppert that if every proper subgroup of G is supersoluble and $|G|$ has at least four distinct prime factors, then G is supersoluble. We shall show by an example that no statement of a similar kind is possible in the situations described in Theorem 1 and the Corollary to Theorem 3.

EXAMPLE 3. *For any integer $n > 1$, there exists a group G such that $|G|$ has n distinct prime factors, every proper self-normalizing subgroup of G is cyclic, and G is not supersoluble.*

Let $G = H \times K$, where H is a tetrahedral group and K a cyclic group of order $p_3 p_4 \dots p_n$, where the p_i are primes such that $5 \leqslant p_3 < p_4 < \dots < p_n$. The only non-nilpotent subgroups of G are the subgroups containing H, and since G/H is cyclic, these are all normal in G. So the only proper self-normalizing subgroups of G are actually its Carter subgroups, which are cyclic of order $3p_3 p_4 \dots p_n$. Since H is not supersoluble, G cannot be supersoluble.

4. *Another sufficient condition for solubility*

In Example 1, we may note that the group G has maximal subgroups not of prime-power index in G, namely its Sylow 2-subgroups. On the other hand, it is a classical theorem of Galois that in a finite soluble group, every maximal subgroup has prime-power index. These observations suggest the following question:

> If a finite group G has each of its abnormal maximal subgroups supersoluble and of prime-power index in G, is G necessarily soluble?

We shall answer this in the affirmative. We need two preliminary lemmas.

LEMMA 4. *Suppose the finite group G has a direct product decomposition $G = G_1 \times \dots \times G_n$. Let ϖ be a set of primes. If G has a Hall ϖ-subgroup H, then $H_i = H \cap G_i$ is a Hall ϖ-subgroup of G_i $(i = 1, \dots, n)$ and $H = H_1 \times \dots \times H_n$.*

Proof. It is enough to consider the case $n = 2$. H_i is a ϖ-subgroup; and $|G_i : H_i| = |HG_i : H|$, from which it follows that H_i is a Hall ϖ-subgroup of G_i. Write $J = H_1 H_2 \leqslant H$. Then

$$\begin{aligned} |G : J| &= |G_1 G_2 : G_1 H_2| . |G_1 H_2 : H_1 H_2| \\ &= |G_2 : (G_1 H_2) \cap G_2| . |G_1 : G_1 \cap (H_1 H_2)| \\ &= |G_2 : H_2| . |G_1 : H_1|. \end{aligned}$$

So $|G : J|$ is divisible by no prime in ϖ, and therefore $J = H$.

LEMMA 5. (J. Szép and L. Rédei [**16**]). *If a group G is factorizable by two of its proper subgroups H and K, that is $G = HK$, and if $H \cap K$ contains a non-trivial normal subgroup of H (or of K), then G is not simple.*

We include an easy proof here.

Proof. Suppose that there exists $L \lhd H$ with $1 < L \leqslant H \cap K$. Let J be the normal closure of L in G, that is the unique smallest normal subgroup of G containing L. We have

$$\begin{aligned} J &= \{g^{-1} Lg : g \in G\}, \\ &= \{(hk)^{-1} L(hk) : h \in H,\ k \in K\}, \text{ since } G = HK, \\ &= \{k^{-1} Lk : k \in K\} \text{ since } L \lhd H, \\ &\leqslant K. \end{aligned}$$

So we have $J \lhd G$ with $1 < J \leqslant K < G$. G is not simple.

THEOREM 4. *If a finite group G has each of its abnormal maximal subgroups supersoluble and of prime-power index in G, then G is soluble.*

Before beginning the proof of Theorem 4, it is perhaps worth remarking that we shall use frequently the fact that a supersoluble group H has normal Sylow p-subgroup, where p is the largest prime factor of $|H|$. We also use the fact that the derived group of a supersoluble group is nilpotent.

Proof of Theorem 4. Suppose the result false, and let the group G provide a counter-example of least order. Since any quotient of G also has each of its abnormal maximal subgroups supersoluble and of prime-power index, the minimality of G implies that G has a unique minimal normal subgroup, say K. K must be insoluble, and since K is characteristically simple, this means that K is a direct power of a simple non-abelian group.

If M is an abnormal maximal subgroup of G, denote by p_M the largest prime factor of $|M|$. If P is a Sylow p_M-subgroup of M, then by the supersolubility of M, $P \lhd M$. Since G has no non-trivial soluble normal subgroup, $M = N_G(P)$. This implies, as in the proof of Lemma 1, that P is actually a Sylow p_M-subgroup of G. We see that if M_1, M_2 are abnormal maximal subgroups of G, and $p_{M_1} = p_{M_2}$, then, since M_1, M_2 are the normalizers in G of conjugate subgroups of G, M_1, M_2 are themselves conjugate in G. So the conjugacy classes $\mathscr{C}_1, \ldots, \mathscr{C}_r$ of abnormal maximal subgroups of G are in 1–1 correspondence with a set of primes $p_1, \ldots, p_r$ such that for $k = 1, \ldots, r$, $M \in \mathscr{C}_k$ if and only if $M = N_G(P)$ for some Sylow p_k-subgroup P of G, p_k being the largest prime factor of $|M|$. We may suppose the primes ordered so that $p_1 > \ldots > p_r$. If, for each $i = 1, \ldots, r$, we choose $M_i \in \mathscr{C}_i$, we see that p_i does not divide $|M_j|$ for any $i < j$. Consequently p_i divides $|G : M_j|$ for all i, j such that $1 \leqslant i < j \leqslant r$.

By the hypothesis that every abnormal maximal subgroup has prime-power index in G, we must have $r \leqslant 2$: for $|G:M_r|$ is divisible by $p_1, \ldots, p_{r-1}$.

Thus G has at most two conjugacy classes of abnormal maximal subgroups; therefore it has precisely two, for otherwise, as in the proof of Theorem 1, G would have a normal Sylow subgroup, contrary to the fact that G has no non-trivial soluble normal subgroup. Let p be the largest prime factor of $|G|$. If P is a Sylow p-subgroup of G, and M a maximal subgroup of G containing $N_G(P)$ $(<G)$, then $M = N_G(P)$. Let L be an abnormal maximal subgroup of G not conjugate to M. Then, by what we have proved above, p does not divide $|L|$, so that L must be a p-complement of G. Suppose $|G:M| =$ a power of q, prime. Let U be a q-complement of M, and therefore also of G. $U \cap K$ is a q-complement of K, and $L \cap K$ a p-complement of K. Let K_1 be a simple direct factor of K. By Lemma 4, $U \cap K_1$ is a q-complement of K_1, and $L \cap K_1$ a p-complement of K_1. Since $U \cap K_1$, $L \cap K_1$ have coprime indices in K_1, they are permutable and

$$K_1 = (U \cap K_1) . (L \cap K_1).$$

We must have $U \cap K_1 < K_1$ and $L \cap K_1 < K_1$, since U and L are supersoluble.

Therefore, by Lemma 5, $A_1 = U \cap L \cap K_1$ contains no non-trivial normal subgroup of either $U \cap K_1$ or $L \cap K_1$. (*)

Now A_1 is a Hall $\{p, q\}'$-subgroup of K_1, and therefore a p-complement of $U \cap K_1$ and a q-complement of $L \cap K_1$.

If P_1 is a Sylow p-subgroup of $U \cap K_1$, then $P_1 \lhd U \cap K_1$ (since $U \cap K_1$ is supersoluble and p is the largest prime factor of $|U \cap K_1|$). By the remark (*) above, $U \cap K_1$ can have no non-trivial normal p'-subgroup: for such a subgroup would be contained in A_1. Therefore P_1 is the Fitting subgroup of $U \cap K_1$. But since the derived group of a supersoluble group is nilpotent, this implies that $U \cap K_1/P_1$ is abelian. Since $A_1 \cong U \cap K_1/P_1$, A_1 is abelian.

Let t be the largest prime factor of $|L \cap K_1|$. Then, if T_1 is a Sylow t-subgroup of $L \cap K_1$, we have $T_1 \lhd L \cap K_1$.

By the remark (*), $T_1 \nleqslant A_1$, a q-complement of $L \cap K_1$. So we must have $t = q$. Consequently q is not the least prime factor of $|K|$: for if it were, $L \cap K_1$ would be a q-group and K_1 a $\{p, q\}$-group; but by a well known theorem of Burnside [**2**; p. 323], this would make K_1 soluble, which is false.

Now let $A = U \cap L \cap K$, a Hall $\{p, q\}'$-subgroup of K. By Lemma 4 and the fact that A_1 is abelian, A is abelian. Let r be the least prime factor of $|K|$. We know that $p > q > r$. Let R be a Sylow r-subgroup of A. Then R is also a Sylow r-subgroup of $K \lhd G$. It follows that $N_G(R)$ is abnormal in G. $N_G(R) < G$, so that by hypothesis $N_G(R)$ is supersoluble.

Therefore $N_K(R) = K \cap N_G(R)$ is supersoluble, and so has normal r-complement, W say, r being the least prime factor of $|N_K(R)|$. It follows that

$$N_K(R) = R \times W.$$

Since R is abelian, R lies in the centre of $N_K(R)$. Thus K has a Sylow r-subgroup contained in the centre of its normalizer, and by another theorem of Burnside, this implies that K has normal r-complement. This is in contradiction to the fact that K is characteristically simple; and this contradiction establishes the theorem.

We have already made use of the characterization of finite nilpotent groups: G is nilpotent if and only if every maximal subgroup is normal in G; or, equivalently, if and only if G has no abnormal maximal subgroup. A non-nilpotent group G satisfying the conditions of Theorem 1 is almost nilpotent, in the sense that its abnormal structure is severely restricted: G has a single conjugacy class of abnormal maximal subgroups—they are nilpotent, and are in fact the only proper abnormal subgroups of G. Also, if F is the Fitting subgroup of G, then by Theorem 1, G/F is nilpotent.

There is an analogous characterization of finite supersoluble groups, due to Huppert [**9**; Satz 9]: G is supersoluble if and only if every maximal subgroup has prime index in G. A non-supersoluble group G satisfying the conditions of Theorem 4 is in a similar way almost supersoluble. By a recent result of W. Gaschütz [**7**], G has a single conjugacy class of maximal subgroups which do not have prime index in G. We prove finally the following corollary to Theorem 4:

COROLLARY. *Suppose G satifies the conditions of Theorem 4, and let F be the Fitting subgroup of G. Then G/F is supersoluble.*

Proof. We use induction on $|G|$. We may assume that $G > F$. Denote by Φ the Frattini subgroup of G. We have $\Phi \leqslant F$, and by a theorem of Gaschütz [**6**; Satz 10], F/Φ is the Fitting subgroup of G/Φ. If $\Phi > 1$, the result follows by the induction hypothesis. So we may assume that $\Phi = 1$. By another result of Gaschütz [**6**; Satz 15], in a non-nilpotent group G with trivial Frattini subgroup, the intersection of all abnormal maximal subgroups coincides with the centre Z of G. We must have $Z < F$, for otherwise if L/F were a non-trivial nilpotent normal subgroup of G/F, L would be a nilpotent normal subgroup of G with $L > F$. Therefore G has an abnormal maximal subgroup M such that $M \not\geqslant F$. Then $G = MF$. M is supersoluble, by hypothesis, and since $G/F \cong M/M \cap F$, G/F is supersoluble. The result follows by induction.

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Abnormal depth and hypereccentric length in finite soluble groups*

By
JOHN S. ROSE

Let G be any finite group, and H any subgroup of G. We shall associate with G and H a non-negative integer $a(G:H)$, determined by the way in which H lies in G and called the "abnormal depth" of H in G. The study of this function was suggested by Professor P. HALL, to whom I wish to express here my sincere thanks for his kindness and generosity in providing many stimulating ideas.

§ 1. Abnormal depth

The definition of $a(G:H)$ is made in terms of chains of subgroups connecting H to G. We shall call a chain of subgroups $H=H_0 \leqq H_1 \cdots \leqq H_r=G$ *balanced* if, for each $i=1, \ldots, r$, either H_{i-1} is normal in H_i or H_{i-1} is abnormal in H_i; and we refer appropriately to two consecutive subgroups $H_{i-1} \leqq H_i$ as forming a *normal link* or an *abnormal link* of the chain.

We recall that a subgroup B is said to be abnormal in a group A when, for every $a \in A$, it is true that $a \in \langle B, a^{-1} B a \rangle$, the subgroup generated by B and $a^{-1} B a$. Equivalently, B is abnormal in A if and only if the following two conditions are satisfied: (i) every subgroup of A containing B coincides with its own normalizer in A, and (ii) B is not contained in any two distinct conjugate subgroups of A. The most familiar examples of abnormal subgroups are provided by the normalizers of Sylow subgroups in any finite group. Further important examples have been discovered by CARTER [4]. He has proved that any finite soluble group G possesses nilpotent self-normalizing subgroups, and that all such subgroups are conjugate and actually abnormal in G. These subgroups are now called the *Carter subgroups* of G.

Since any chain in which each subgroup is maximal in its successor is balanced, it is always possible to connect a subgroup to a finite group by means of a balanced chain of subgroups. This fact enables us to make the following

Definition. Let H be a subgroup of the finite group G. Then $a(G:H)=$ the least number of abnormal links appearing in any balanced chain of subgroups connecting H to G. $a(G:H)$ is the *abnormal depth* of H in G.

To express the idea underlying this definition briefly, we may say that the number $a(G:H)$ is designed to increase with the complexity of the embedding of H in G. From this point of view, the simplest situation occurs when $a(G:H)=0$,

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and we see that this is precisely when H is subnormal in G. It ought perhaps to be pointed out that, in spite of the use of the word "depth" in this context, a group G may have subgroups H, K such that $H>K$, but $a(G:H)>a(G:K)$.

It is easy to see that if φ is any homomorphism of G, then $a(G^{\varphi}:H^{\varphi})\leqq a(G:H)$. Also if K is a subgroup of H, then $a(G:K)\leqq a(G:H)+a(H:K)$.

The first aim is to establish certain bounds for $a(G:H)$ in particular circumstances of G and H. We may observe at once that if P is any subgroup of prime power order of a finite group G, then $a(G:P)\leqq 1$; for P lies in some Sylow subgroup $\bar{P}$ of G, P is subnormal in $N_G(\bar{P})$, and $N_G(\bar{P})$ is abnormal in G. (For any subgroup H of a group G, $N_G(H)$ denotes the normalizer of H in G.)

We shall now show that when G is a finite soluble group and H a nilpotent subgroup, $a(G:H)$ is bounded in terms of the nilpotent length of G. We need first a definition and a lemma.

Definition. If H is any subgroup of a finite group G, define recursively: $H_0=H$, $H_i=N_G(H_{i-1})$ $(i=1,2,\ldots)$. Then $H_0\leqq H_1\leqq H_2\leqq\ldots$, and since G is finite, this ascending chain of subgroups must become stationary, in a subgroup which we denote by $N_G^{\infty}(H)$. This is called the *hypernormalizer* of H in G. We see that, by this definition, H is subnormal in $N_G^{\infty}(H)$ and $N_G^{\infty}(H)$ is self-normalizing in G.

Lemma 1. *Suppose that $G=HK$, where H, K are nilpotent subgroups of the finite soluble group G, and $K\trianglelefteqq G$. Then the hypernormalizer $H^*=N_G^{\infty}(H)$ of H in G is a Carter subgroup of G.*

Proof. From $H\leqq H_1=N_G(H)\leqq HK$, it follows that $H_1=H(H_1\cap K)$. $H\trianglelefteqq H_1$, by definition of H_1, and $H_1\cap K\trianglelefteqq H_1$, since $K\trianglelefteqq G$; moreover, H and $H_1\cap K$ are both nilpotent. Therefore, by FITTING's theorem, H_1 is nilpotent. Also $G=H_1K$. Repeated application of this argument shows that H^* is nilpotent. H^* is, however, self-normalizing in G, and so H^* is a Carter subgroup of G.

We are now able to establish

Theorem 1. *If G is a finite soluble group, of nilpotent length n, and H is a nilpotent subgroup of G, then $a(G:H)\leqq n-1$.*

Proof. There exists, by hypothesis, a series of normal subgroups of G: $1=U_0<U_1<\cdots<U_n=G$, such that each quotient U_i/U_{i-1} is nilpotent $(i=1,\ldots,n)$. Define

$$H_i=HU_i \qquad (i=0,1,\ldots,n),$$

$$N_i=N_{H_{i+1}}^{\infty}(H_i) \qquad (i=0,\ldots,n-1).$$

Since G/U_{n-1} is nilpotent, every subgroup of G containing H_{n-1} is subnormal in G, and therefore $N_{n-1}=G$.

We apply Lemma 1 to H_{i+1}/U_i $(i=0,\ldots,n-1)$. This is possible because $H_{i+1}/U_i=H_i/U_i\cdot U_{i+1}/U_i$, with $H_i/U_i\cong H/H\cap U_i$, which is nilpotent, U_{i+1}/U_i is nilpotent and $U_{i+1}/U_i\trianglelefteqq H_{i+1}/U_i$. We conclude that $N_{H_{i+1}/U_i}^{\infty}(H_i/U_i)$ is a

Carter subgroup of H_{i+1}/U_i. But $N^{\infty}_{H_{i+1}/U_i}(H_i/U_i)=N_i/U_i$, and so we see that N_i is abnormal in H_{i+1} $(i=0, \ldots, n-1)$. Thus, if in the chain of subgroups

$$H=H_0 \leqq N_0 \leqq H_1 \leqq N_1 \leqq \cdots \leqq H_{n-1} \leqq N_{n-1}=G,$$

we insert normal links between H_i and N_i for $i=0, \ldots, n-1$ – as we may, since H_i is subnormal in N_i – then we obtain a balanced chain connecting H to G. It is clear that this chain has not more than $n-1$ abnormal links, and this yields the result.

Theorem 1 shows in particular that if H is a nilpotent subgroup of a finite metanilpotent group G (that is, G has nilpotent length $\leqq 2$), then $a(G:H) \leqq 1$. We ask whether this statement remains true if the condition on H is removed. In answer we shall prove two results, which display strongly contrasting behaviour in two different classes of metanilpotent groups.

Theorem 2. *If G is any finite abelian-by-nilpotent group, then every subgroup H of G satisfies $a(G:H) \leqq 1$.*

Theorem 3. *For any positive integer n, there exists a finite supersoluble group G with a subgroup H such that $a(G:H)=n$.*

(We recall that the derived group of a supersoluble group is nilpotent, so that a supersoluble group is nilpotent-by-abelian and, in particular, metanilpotent.)

We begin by deducing Theorem 2 from Theorem 1.

Proof of Theorem 2. We use induction on the order of G. The hypothesis is that G has an abelian normal subgroup A such that G/A is nilpotent. The subgroup HA of G also has A as an abelian normal subgroup, and HA/A is a subgroup of G/A and therefore nilpotent. Hence if $HA<G$, the induction hypothesis implies that $a(HA:H) \leqq 1$. Since G/A is nilpotent, HA is subnormal in G, and so $a(G:H) \leqq a(HA:H) \leqq 1$. Therefore we may suppose that $HA=G$. Then $H \cap A \trianglelefteqq H$, since $A \trianglelefteqq G$, and $H \cap A \trianglelefteqq A$, since A is abelian; so that $H \cap A \trianglelefteqq G$. Then $G/H \cap A$ has $A/H \cap A$ as an abelian normal subgroup, with quotient group isomorphic to G/A, which is nilpotent. Thus if $H \cap A>1$, the induction hypothesis implies that $a(G/H \cap A: H/H \cap A) \leqq 1$. It is clear that $a(G/H \cap A: H/H \cap A)=a(G:H)$, and so $a(G:H) \leqq 1$. Therefore we may suppose further that $H \cap A=1$. This implies that $H \cong HA/A=G/A$, so that H is nilpotent. In this case however, Theorem 1 shows that $a(G:H) \leqq 1$. The result follows by induction.

§ 2. The direct square of a group, and its diagonal subgroup

In preparation for the construction needed to establish Theorem 3, we shall investigate the structure of a certain sublattice of the lattice of subgroups of $G \times G$, the direct square of an arbitrary group G. The results contained in Lemma 2 are of an elementary nature, but might perhaps prove helpful in other contexts. There are no finiteness restrictions on the groups appearing in this section.

We begin with some notation and terminology. For any elements x, y of a group G, $[x, y]=x^{-1}y^{-1}xy$. For any subgroups H, K of G, $[H, K]$ is the subgroup of G generated by all elements $[h, k]$ with $h \in H$, $k \in K$. $[G, G]=G'$, the derived group of G. By a *factor* of a group G, we shall mean a group H/K, where $H \trianglelefteqq G$, $K \trianglelefteqq G$ and $H \geqq K$. H/K is called a *central* factor of G if $[G, H] \leqq K$. H/K is called a *hypercentral* factor of G if there exists a chain of subgroups $H=H_0 \geqq H_1 \geqq \cdots \geqq H_r=K$ such that each $H_i \trianglelefteqq G$ and each H_{i-1}/H_i is a central factor of G $(i=1, \ldots, r)$. A factor H/K is called a *hypereccentric* factor of G if there is no central factor H_1/K_1 of G such that $H \geqq H_1 > K_1 \geqq K$.

Some comment on the last definition is perhaps needed. When the group G possesses a chief series, and in particular when G is finite, the notion of a hypereccentric factor of G as defined here coincides with that as defined elsewhere, for instance by HALL [6]. SCHREIER's theorem serves to show this. The present definition is however applicable also to groups which do not possess chief series. It is preferred as being more conveniently adapted to the statement (iv) of Lemma 2 below.

For any given group G, we shall denote by G^* the direct square of G, that is the direct product of 2 copies of G:

$G^*=$ the set of all pairs (g_1, g_2) with $g_1, g_2 \in G$, and multiplication by components.

We define the *diagonal subgroup* of G^* as

$$\hat{G}=\text{the set of all pairs } (g, g) \text{ with } g \in G.$$

It is clear that $\hat{G}$ is a subgroup of G^*, and that $\hat{G} \cong G$.

We shall characterize the lattice of all subgroups of G^* containing $\hat{G}$. For this purpose, we introduce the following notation. If K is any subgroup of G, let

$$K_1=\text{the set of all pairs } (k, 1) \text{ with } k \in K,$$

$$K_2=\text{the set of all pairs } (1, k) \text{ with } k \in K.$$

K_1 and K_2 are subgroups of G^*, both isomorphic to K. If $K \trianglelefteqq G$, then $K_1 \trianglelefteqq G^*$, $K_2 \trianglelefteqq G^*$, and $K_1\hat{G}=K_2\hat{G}$: for if $k \in K$, then $(1, k)=(k^{-1}, 1)(k, k)$, so that $K_2 \leqq K_1\hat{G}$, and similarly $K_1 \leqq K_2\hat{G}$.

When $K \trianglelefteqq G$, let

$$K^*=K_1\hat{G}=K_2\hat{G}.$$

We note that this gives a consistent notation, since $G_1\hat{G}=G_2\hat{G}=G^*$, as defined originally; and also that $\hat{G}=1^*$. The correspondence $K \to K^*$ defines a mapping $*$ of the lattice of normal subgroups of G into the lattice of subgroups of G^* containing $\hat{G}$. The properties of this mapping are contained in the following

Lemma 2. *The mapping* $*$: $K \to K^*$ *of the lattice of normal subgroups of* G *to the lattice of subgroups of* G^* *containing* $\hat{G}$ *is bijective, and a lattice-iso-*

morphism. The inverse image of a subgroup K^* of G^* containing $\hat{G}$ is the subgroup K of G corresponding in the natural isomorphism $G_1 \cong G$ to the subgroup $K_1 = G_1 \cap K^*$ of G_1 (or, equivalently, corresponding in $G_2 \cong G$ to $K_2 = G_2 \cap K^*$). Furthermore,

(i) *Any two subgroups of G^* containing $\hat{G}$ are permutable; no two are conjugate in G^*.*

(ii) *K^* is normal in H^* if and only if H/K is a central factor of G.*

(iii) *K^* is subnormal in H^* if and only if H/K is a hypercentral factor of G.*

(iv) *K^* is abnormal in H^* if and only if H/K is a hypereccentric factor of G.*

Proof. We show first that $*$ is injective. Suppose $K \trianglelefteqq G$, $H \trianglelefteqq G$ and $K^* = H^*$, that is $K_1 \hat{G} = H_1 \hat{G}$. Then for any $k \in K$, $(k, 1) = (h, 1)\,(g, g)$ for some $h \in H$ and $g \in G$. It follows from this equation that $k = h$. Thus $K \leqq H$, and similarly $H \leqq K$. Therefore $K = H$.

Next we show that $*$ is surjective. Given a subgroup $\overline{K}$ of G^* containing $\hat{G}$, let K be the subgroup of G corresponding in the natural isomorphism $G_1 \cong G$ to the subgroup $G_1 \cap \overline{K}$ of G_1. Then $G_1 \cap \overline{K} = K_1$. Since $G_1 \trianglelefteqq G^*$, it follows that $K_1 \trianglelefteqq \overline{K}$, and therefore K_1 is normalized by $\hat{G}$. Hence, for any $k \in K$ and $g \in G$,

$$(g, 1)^{-1}\,(k, 1)\,(g, 1) = (g^{-1}\,k\,g, 1) = (g, g)^{-1}\,(k, 1)\,(g, g) \in K_1 .$$

Thus $K_1 \trianglelefteqq G_1$, and so $K \trianglelefteqq G$. Moreover, since $\hat{G} \leqq \overline{K} \leqq G^* = G_1\,\hat{G}$, it follows that $\overline{K} = (G_1 \cap \overline{K})\,\hat{G} = K_1\,\hat{G}$. Hence $\overline{K} = K^*$. This proves that $*$ is surjective, and also that the inverse image of K^* is K, where $K_1 = G_1 \cap K^*$.

We want to show also that for any $K \trianglelefteqq G$, $K_2 = G_2 \cap K^*$. We have observed already that $K_2 \leqq K^*$. Any element of K^* may be expressed as $(1, k)\,(g, g)$ for some $k \in K$ and $g \in G$, and this belongs to G_2 only if $g = 1$. Hence $G_2 \cap K^* \leqq K_2$, and so $K_2 = G_2 \cap K^*$.

We have proved that $*$ is a bijective mapping, and it is clear that $H^* \leqq K^*$ if and only if $H \leqq K$. In order to prove that $*$ is a lattice-isomorphism, it remains only to show that $*$ preserves joins and intersections: if $H \trianglelefteqq G$, $K \trianglelefteqq G$, then $\langle H, K\rangle^* = \langle H^*, K^*\rangle$ and $(H \cap K)^* = H^* \cap K^*$. It is easy to verify this directly; or we may refer to the fact that any bijective mapping $*\colon \mathscr{L} \to \mathscr{L}^*$ from a lattice $\mathscr{L}$ onto a lattice $\mathscr{L}^*$, such that $X^* \leqq Y^*$ if and only if $X \leqq Y$, for any $X, Y \in \mathscr{L}$, is necessarily a lattice-isomorphism: see for instance Birkhoff [*1*], pp. 20, 21.

(i) We see at once from the definition of $*$ that any H^* and K^* are permutable. Now suppose that $K^* = (H^*)^x$ for some $x \in G^*$. Then $G_1 \cap K^* = G_1^x \cap (H^*)^x = (G_1 \cap H^*)^x = G_1 \cap H^*$, since $G_1 \cap H^* = H_1 \trianglelefteqq G^*$. Hence $K^* = (G_1 \cap K^*)\,\hat{G} = (G_1 \cap H^*)\,\hat{G} = H^*$.

(ii) Suppose $K^* \trianglelefteqq H^*$, that is $K_1\,\hat{G} \trianglelefteqq H_1\,\hat{G}$. For any $g \in G$ and $h \in H$, $(h, 1)^{-1}\,(g, g)\,(h, 1) \in K_1\,\hat{G} = \hat{G}\,K_1$, so that $(h^{-1}\,g\,h, g) = (g'\,k, g')$ for some

$k \in K$ and $g' \in G$. Then $g = g'$ and $h^{-1} g h = g k$, so that $[g, h] \in K$. Hence $[G, H] \leqq K$.

Conversely, suppose $[G, H] \leqq K$. We want to show that $K_1 \hat{G} \trianglelefteq H_1 \hat{G}$, and for this purpose, since $K_1 \trianglelefteq G^*$, it is enough to show that for any $g \in G$ and $h \in H$, $(h, 1)^{-1}(g, g)(h, 1) \in K_1 \hat{G}$. By hypothesis, $[g, h] \in K$, so that $h^{-1} g h = g k$ for some $k \in K$, and therefore $(h^{-1} g h, g) \in \hat{G} K_1 = K_1 \hat{G}$, as we need.

(iii) This follows immediately from (ii).

(iv) In view of (i), no K^* is contained in two distinct conjugate subgroups of G^*. Therefore K^* is abnormal in H^* if and only if, for every J^* such that $K^* \leqq J^* \leqq H^*$, $N_{H^*}(J^*) = J^*$. By (ii), $N_{G^*}(J^*) = Z_J^*$, where Z_J/J is the centre of G/J. Thus K^* is abnormal in H^* if and only if, for every J^* such that $K^* \leqq J^* \leqq H^*$, $Z_J^* \cap H^* = J^*$; or equivalently, by the lattice-isomorphism, if and only if, for every $J \trianglelefteq G$ such that $K \leqq J \leqq H$, $Z_J \cap H = J$. This is true if and only if there is no central factor A/B of G such that $K \leqq B < A \leqq H$; that is, if and only if H/K is hypereccentric in G.

§ 3. Hypereccentric length

If G is any finite group, there exist series of normal subgroups of G, $1 = G_0 \leqq G_1 \leqq \cdots \leqq G_r = G$, in which each factor G_i/G_{i-1} is either hypercentral or hypereccentric in G: indeed, any chief series of G is of this kind. We shall call such a series *separated*.

Definition. Let G be a finite group. Then $\eta(G)$ = the least number of hypereccentric factors appearing in any separated series of G. We shall call $\eta(G)$ the *hypereccentric length* of G.

We observe that $\eta(G) = 0$ if and only if G is nilpotent.

CARTER [3] has shown that if G is a finite soluble group, of nilpotent length n, and if $G = L_0 > L_1 > \cdots > L_n = 1$ is the lower nilpotent series of G, then L_i/L_i' is a hypereccentric factor of G for $1 \leqq i \leqq n-1$. Thus if $L_i' = L_{i+1}$ for all i such that $1 \leqq i \leqq n-1$, then $L_1/1$ is a hypereccentric factor of G, and so $\eta(G) \leqq 1$. In particular, if the lower nilpotent series of G coincides with its derived series (as happens if G is an A-group, that is a finite soluble group in which every Sylow subgroup is abelian), then $\eta(G) \leqq 1$. Also, if G is an abelian-by-nilpotent group, then $\eta(G) \leqq 1$.

Next we note a connexion between abnormal depth and hypereccentric length, which is the key to the proof of Theorem 3.

Lemma 3. *If G is any finite group, G^* the direct square of G, and $\hat{G}$ the diagonal subgroup of G^*, then $a(G^* : \hat{G}) = \eta(G)$.*

Proof. This follows directly from Lemma 2, (iii) and (iv).

We see from Lemma 3 that, in order to prove Theorem 3, it will be enough to prove

Theorem 3′. *For any positive integer n, there exists a finite supersoluble group G with $\eta(G) = n$.*

Once this has been achieved, we can form the direct square G^* of G; G^* will also be finite and supersoluble, and will have a subgroup $\hat{G}$ (its diagonal subgroup) such that $a(G^*:\hat{G})=n$.

Any finite group G has an "upper separated series". (We may compare this with the upper p-series of a p-soluble group, and hypereccentric length with p-length: see HALL and HIGMAN [7].) In order to define this, we note that G has a unique greatest normal subgroup H such that $H/1$ is a hypercentral factor of G, namely H is the hypercentre of G; and also that G has a unique greatest normal subgroup E such that $E/1$ is a hypereccentric factor of G. To prove the latter statement, suppose that $E_1 \trianglelefteq G$, $E_2 \trianglelefteq G$ and $E_1/1$, $E_2/1$ are both hypereccentric in G. We want to show that $E_1 E_2/1$ is hypereccentric in G. If this were false, there would be a central chief factor A/B of G such that $E_1 \leqq B < A \leqq E_1 E_2$, since $E_1/1$ is hypereccentric in G. Then $A=E_1(A \cap E_2)$ and $B=E_1(B \cap E_2)$. But $[G, A] \leqq B$ would imply that $[G, A \cap E_2] \leqq B \cap E_2$, so that $A \cap E_2/B \cap E_2$ would be a central factor of G. The supposition that $E_2/1$ is hypereccentric in G would then imply that $A \cap E_2 = B \cap E_2$, and therefore that $A=B$, a contradiction.

We shall call E the *shell* of G. We define the *upper separated series* of G:

$$(\dagger) \qquad 1=E_0 \leqq H_0 \leqq E_1 \leqq H_1 \leqq E_2 \leqq \cdots$$

recursively by

$$E_0=1,\ H_i/E_i = \text{the hypercentre of } G/E_i,$$

$$E_{i+1}/H_i = \text{the shell of } G/H_i \qquad (i=0,1,2,\ldots).$$

It may readily be shown that if

$$1=G_0 \leqq G_1 \leqq G_2 \leqq \cdots \leqq G_{2r} \leqq G_{2r+1}=G$$

is any separated series of G, in which G_{2i+1}/G_{2i} is hypercentral in G for $i=0,1,\ldots,r$, and G_{2i}/G_{2i-1} is hypereccentric in G for $i=1,\ldots,r$, then $G_{2i} \leqq E_i$ and $G_{2i+1} \leqq H_i$ for $i=0,1,\ldots,r$. Hence in the series (†), $H_n=G$ for $n=\eta(G)$; and, by definition of $\eta(G)$, $H_{n-1} \neq G$. The upper separated series of G, terminated at H_n for the least integer n such that $H_n=G$, is thus a separated series of G in which the number of hypereccentric factors is least possible.

When G is soluble, and $n=\eta(G)$, $E_n < H_n = G$, for no non-trivial quotient of G can be hypereccentric in G.

It is possible to define also a "lower separated series" of G, but we shall not refer to that in what follows.

The construction which will be used to establish Theorem 3′ is made by means of an extension of a group of unitriangular matrices. Let m be any given positive integer, and let K be a field. Denote by e_{ij} the m-square matrix, with entries in K, which has the identity element 1 of K in the i-th row and j-th column, and the zero element 0 of K everywhere else. A matrix of the form

$$1+\sum_{i<j} \lambda_{ij} e_{ij},$$

where 1 now denotes the identity m-square matrix, and each $\lambda_{ij} \in K$, is called *unitriangular*. Such a matrix is invertible, and thus belongs to the general linear group $GL_m(K)$. The set U of all unitriangular matrices forms a subgroup of $GL_m(K)$; and when K is a finite field, of characteristic p, U is a Sylow p-subgroup of $GL_m(K)$. The structure of U, when K is finite and p an odd prime, has been investigated by WEIR [8]. The following facts will be needed:

(i) U is generated by all elements of the form

$$1+\lambda e_{i,i+1} \qquad (\lambda \in K, i=1,2,\dots,m-1).$$

(ii) Let L_k be the subset of U consisting of all elements

$$1+\sum_{i<j} \lambda_{ij} e_{ij}$$

for which $\lambda_{ij}=0$ whenever $j-i<k$ $(1 \leqq k \leqq m)$. Then L_k is a subgroup of U, and the lower central series of U coincides with $U=L_1>L_2>\cdots>L_m=1$. This series also coincides with the upper central series $1=Z_0<Z_1<\cdots<Z_{m-1}=U$ of U; that is, $Z_k=L_{m-k}$ for $0 \leqq k \leqq m-1$.

For the purpose of proving Theorem 3′, we choose $m=2n$, where n is any given positive integer, and $K=GF(p)$, a Galois field with p elements, where p is an odd prime. We consider the subgroup $G=\langle U, t\rangle$ of $GL_{2n}(p)$, where

$$t=\operatorname{diag}\{-1,+1,-1,+1,\dots,-1,+1\}=\sum_{i=1}^{2n}(-1)^i e_{ii}.$$

It is clear that t has order 2 and normalizes U, so that G is a split extension of U by t. The operation of t on U is determined by the relation $t^{-1} e_{ij} t=(-1)^{i+j} e_{ij}$.

We shall show that $\eta(G)=n$. Each term Z_k $(0 \leqq k \leqq 2n-1)$ of the upper central series of U is a characteristic subgroup of U, and therefore a normal subgroup of G. We set $Z_{2n}=G$, and consider the series:

$$1=Z_0<Z_1<\cdots<Z_{2n-1}=U<Z_{2n}=G.$$

We shall show that this is the upper separated series of G, and that its first non-trivial factor $Z_1/1$ is hypereccentric; hence that $\eta(G)=n$.

We begin by proving that Z_k/Z_{k-1} *is hypereccentric in G when k is odd, and hypercentral in G when k is even*. Let A/B be any chief factor of G for which $Z_{k-1} \leqq B<A \leqq Z_k$, where k is any fixed positive integer $<2n$. Then, since Z_k/Z_{k-1} is a central factor of U, A/B is certainly central in U. Therefore A/B is central in G if and only if t induces the identity automorphism in A/B. Let a be any element of A; and set $u_i=1+e_{i,i+2n-k}$ for $i=1,\dots,k$. We see that Z_k/Z_{k-1} is the direct product of k groups of order p, generated by the elements $Z_{k-1} u_i$ $(1 \leqq i \leqq k)$. Then since $Z_{k-1} \leqq B$ and $A \leqq Z_k$, we have $Ba=Bu_1^{r_1} u_2^{r_2} \dots u_k^{r_k}$, for certain integers $r_1, r_2, \dots, r_k$. Now t transforms the element u_i of G to $u_i^t= 1+(-1)^k e_{i,i+2n-k}$, so that $u_i^t=u_i$ if k is even, and $u_i^t=u_i^{-1}$ if k is odd. Therefore, since $u_1, u_2, \dots, u_k$ commute modulo B, t transforms Ba to Ba

if k is even, and to $B a^{-1}$ if k is odd. Thus when k is even, t certainly induces the identity automorphism in A/B; but when k is odd, and a is an element outside B, a could only be fixed by t if $a^2 \in B$, which would, however, since B is a p-group and p an odd prime, imply that $a \in B$, a contradiction. Hence t induces the identity automorphism in A/B if and only if k is even. Since it is obvious that Z_{2n}/Z_{2n-1} is central and therefore hypercentral in G, the assertion at the beginning of the paragraph is proven.

The next step is to note that *every proper normal subgroup of G is a p-group, and so lies in U*. In order to prove this, it suffices to show that no proper normal subgroup of G contains t. This follows from the fact that, for any $\lambda \in GF(p)$ and any $i = 1, 2, \ldots, 2n-1$,

$$\left[1-\frac{\lambda}{2}\, e_{i,\,i+1}, t\right] = \left(1+\frac{\lambda}{2}\, e_{i,\,i+1}\right)\left(1+\frac{\lambda}{2}\, e_{i,\,i+1}\right) = 1+\lambda\, e_{i,\,i+1}.$$

(Division by 2 is permissible, since p is odd.) Since, by (i) above, U is generated by all elements of the form $1+\lambda\, e_{i,\,i+1}$, and since any normal subgroup of G containing t must also contain all elements of the form $[g, t]$ with $g \in G$, the assertion follows.

The final argument in proving $\eta(G)=n$ is to show that $Z_{2i+1}/Z_{2i} =$ *the shell of* G/Z_{2i}, $Z_{2i+2}/Z_{2i+1} =$ *the hypercentre of* G/Z_{2i+1}, *for* $i = 0, 1, \ldots, n-1$. Suppose that $1 = E_0 \leqq H_0 \leqq E_1 \leqq H_1 \leqq E_2 \leqq \cdots$ is the upper separated series of G. We observe first that $H_0 = 1$: for otherwise G would have a non-trivial centre, V say, and, by what was proved in the preceding paragraph, $V \leqq U$; but this would imply that $V \leqq Z_1 =$ centre of U, and this is impossible since we have shown already that $Z_1/1$ is hypereccentric in G. Next, since $H_0 = 1$ and $Z_1/1$ is hypereccentric in G, $Z_1 \leqq E_1$. Since G is soluble, $E_1 < G$ and therefore $E_1 \leqq U$. If $Z_1 < E_1$, E_1/Z_1 would be a non-trivial normal subgroup of the p-group U/Z_1, and so E_1/Z_1 would have non-trivial intersection with the centre of U/Z_1, namely Z_2/Z_1; this is however impossible, since Z_2/Z_1 is hypercentral and E_1/Z_1 hypereccentric in G. Hence $E_1 = Z_1$.

We now prove by induction on k that $Z_{2k} = H_k$, $Z_{2k+1} = E_{k+1}$ for $k = 0, 1, \ldots, n-1$. These equalities have been established for $k = 0$. Suppose now that they are true for $k = i-1$, where $0 < i < n$. Then Z_{2i}/Z_{2i-1} is hypercentral in G, so that $Z_{2i} \leqq H_i$. $H_i < G$, since $i < n$ and therefore G/Z_{2i} is not hypercentral in G. Therefore $H_i \leqq U$. If $Z_{2i} < H_i$, H_i/Z_{2i} would have non-trivial intersection with Z_{2i+1}/Z_{2i} (by the same argument as for E_1/Z_1 with Z_2/Z_1), which is impossible since Z_{2i+1}/Z_{2i} is hypereccentric and H_i/Z_{2i} hypercentral in G. Hence $Z_{2i} = H_i$. From this it follows that $Z_{2i+1} \leqq E_{i+1}$. Moreover $E_{i+1} < G$, because G is soluble and $H_i \neq G$. Therefore $E_{i+1} \leqq U$. Application of the same argument as before yields $Z_{2i+1} = E_{i+1}$. This completes the induction argument.

In this way we have shown that G is a group of hypereccentric length n. It is clear that G is metanilpotent, but it is claimed that G is even supersoluble. This may be proved directly quite easily; or it may be deduced from the following

Lemma 4. *If a group G is an extension of a finite nilpotent group H by a group of order 2, then G is supersoluble.*

Proof. We may suppose $H \neq 1$. Let $1 = Z_0 < Z_1 < \cdots < Z_r = H$ be the upper central series of the nilpotent group H. Each $Z_i \lhd G$. We may therefore refine the series $1 = Z_0 < Z_1 < \cdots < Z_r < G$ to a chief series of G. Let A/B be any chief factor of G belonging to this series, and such that $A \leqq H$. Then A/B is a central factor of H, and so A/B becomes a right G/H-module by defining, for any $a \in A$ and $g \in G$, $(B\,a)^{Hg} = B(g^{-1}\,a\,g)$. Since A/B is a chief factor of G, A/B is an irreducible G/H-module. It follows from Lemma 5 below, with $n = 2$, that A/B is a cyclic group. Hence G is supersoluble.

Lemma 5. *Suppose that C is a cyclic group of finite order $n > 1$, and that M is an irreducible C-module. Then M is generated, as an abelian group, by a set of $< n$ elements.*

Proof. For convenience, the group structure of M is written additively. Choose $m \in M$, $m \neq 0$, and suppose that c generates C. The elements m, mc, $m\,c^2, \ldots, m\,c^{n-1}$ generate a subgroup L of M which is obviously C-invariant, that is, which is a submodule of M. Since M is irreducible and $L \neq 0$, $L = M$. Let $m^* = m + mc + mc^2 + \cdots + mc^{n-1}$. Then $m^*\,c = m^*$. If $m^* \neq 0$, then m^* generates a C-invariant non-trivial subgroup of M, which is therefore the whole group M; M is then a cyclic group, and the result is true. Otherwise $m^* = 0$, and at least one of the generators m, mc, $mc^2, \ldots, mc^{n-1}$ of M is superfluous; that is, the abelian group M has a set of $< n$ generators.

§ 4. Nilpotent subabnormal subgroups

No example has yet been given of a soluble group with a nilpotent subgroup of abnormal depth > 1, and so an example will be supplied in this section. However, the question of whether Theorem 1 gives a best possible result for $n > 2$ remains unanswered.

Some remarks on nilpotent subabnormal subgroups are pertinent to the example. A subgroup H is said to be *subabnormal* in a group G if there exists a chain of subgroups $H = H_0 \leqq H_1 \leqq \cdots \leqq H_r = G$, connecting H to G, such that H_{i-1} is abnormal in H_i, for each $i = 1, \ldots, r$. A finite group must possess minimal subabnormal subgroups, and these are necessarily nilpotent. It is a remarkable fact, discovered by HALL [6, Theorem 4.8], that in a finite soluble group G, the minimal subabnormal subgroups form a single conjugacy class; they are actually the system normalizers of G.

Let G be a finite soluble group, and denote by $\mathcal{N}$ the set of all its nilpotent subabnormal subgroups. Then $\mathcal{N}$ has a single conjugacy class in G of minimal members, namely the system normalizers of G. The Carter subgroups of G also belong to $\mathcal{N}$, and form a conjugacy class in G of maximal members of $\mathcal{N}$. If G is metanilpotent, then by a theorem of CARTER [2, Theorem 5.6], the system normalizers and Carter subgroups coincide, so that in this case $\mathcal{N}$ consists of a single class of conjugate subgroups of G. If G has nilpotent length 3, $\mathcal{N}$

may contain more than one class of conjugate subgroups of G, but a more recent result of CARTER [5, Theorem 3] shows that the Carter subgroups of G are the only maximal members of $\mathcal{N}$. This leads to the following Corollary of Theorem 1.

Corollary. *If G is a finite soluble group, of nilpotent length $n \geqq 3$, and H is a nilpotent subabnormal subgroup of G, then $a(G:H) \leqq n-2$.*

Proof. Let $G=L_0>L_1>\cdots>L_n=1$ be the lower nilpotent series of G. HL_3/L_3 is a nilpotent subabnormal subgroup of G/L_3, and since G/L_3 has nilpotent length 3, it follows that HL_3/L_3 lies in a Carter subgroup R/L_3 of G/L_3. HL_3 is subnormal in R, and R is abnormal in G, so that $a(G:HL_3) \leqq 1$. But HL_3 is a group of nilpotent length $\leqq n-2$, and so by Theorem 1, $a(HL_3:H) \leqq n-3$. Hence

$$a(G:H) \leqq a(G:HL_3) + a(HL_3:H) \leqq n-2 .$$

It might be asked whether, for any finite soluble group G, the Carter subgroups of G are the only maximal members of the set $\mathcal{N}$. We shall show that this is false by giving an example of a group G of nilpotent length 4, with a maximal nilpotent subgroup V, which is subabnormal in G but not a Carter subgroup of G; and we shall see that $a(G:V)=2$.

Example. G is the wreath product of a cyclic group of order 5 by a symmetric group of degree 4, taken with respect to the natural representation. This group has been described already, by CARTER [5, pp. 562, 563], and we adopt his notation (except that we have A instead of N, to avoid confusion with normalizers). $G=A\overline{G}$ and $A \cap \overline{G}=1$, where A is elementary abelian of order 5^4, generated by elements a_1, a_2, a_3, a_4 of order 5; $\overline{G}$ is the symmetric group on the set $\{1, 2, 3, 4\}$; and for each $x \in \overline{G}$, $1 \leqq i \leqq 4$, $x^{-1} a_i x = a_{ix}$.

$\overline{E}=\langle(1\ 3\ 2\ 4), (1\ 2)\rangle$ is a Carter subgroup of $\overline{G}$, and it is easy to show that $Z\overline{E}$ is a Carter subgroup of G, where $Z=\langle a_1 a_2 a_3 a_4\rangle$ = the centre of G. Thus the Carter subgroups of G have order 5.2^3.

Consider now $\overline{D}=\langle(1\ 2)\rangle$. The centralizer of $\overline{D}$ in A is $K=\langle a_1 a_2, a_3, a_4\rangle$. Let $V=\langle K, \overline{D}\rangle = K \times \overline{D}$, which is nilpotent.

$$N_G(V)=N_G(K) \cap N_G(\overline{D}) .$$

$$N_G(\overline{D})=N_A(\overline{D}) \cdot N_{\overline{G}}(\overline{D}) = K \cdot \langle(1\,2), (3\,4)\rangle \leqq N_G(K) .$$

Hence $N_G(V)=N_G(\overline{D})=K \cdot \langle(1\ 2), (3\ 4)\rangle$. Thus V has index 2 in $N_G(V)$, and $N_G(V)$ is non-nilpotent. Hence V is a maximal nilpotent subgroup of G. V has order $5^3.2$, and so is not a Carter subgroup of G. However, V is subabnormal in G, for $V=K\overline{D}$ has index 5 in $A\overline{D}$, so that V is maximal in $A\overline{D}$. But V is not normal in $A\overline{D}$, since $a_1^{-1}(1\ 2)\, a_1 = a_1^{-1} a_2 (1\ 2) \notin V$, and so V is abnormal in $A\overline{D}$. $A\overline{D}$ is subabnormal in G, since in the isomorphism $G/A \cong \overline{G}$, $A\overline{D}/A$ corresponds to $\overline{D}$, and $\overline{D}$ is subabnormal in $\overline{G}$. ($\overline{D}$ is in fact a system normalizer of $\overline{G}$.)

Finally, $a(G:V)=2$. In order to establish this, we remark first that V is contained in no proper normal subgroup of G; this is easily shown to be true for any subabnormal subgroup of a group. Next, it is well known that the normalizer of a maximal nilpotent subgroup is necessarily self-normalizing in a group. Hence, since also V is a maximal subgroup of $N_G(V)$, we could have $a(G:V)=1$ only if $N_G(V)$ were abnormal in G. However, $N_G(V)<A\cdot\langle(1\,2),(3\,4)\rangle\lhd A\bar{E}<G$, and so $N_G(V)$ is not abnormal in G. Therefore $a(G:V)>1$. Since $N_G(V)$ is maximal, and therefore abnormal, in $A\cdot\langle(1\,2),(3\,4)\rangle$, and $A\bar{E}$ is abnormal in G, $a(G:V)=2$.

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Dept. of Math., The University, Newcastle upon Tyne, England

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ON A SPLITTING THEOREM OF GASCHÜTZ

by JOHN S. ROSE

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1. Let G be any finite group, and p any prime number. (All groups to be considered here are finite, and we assume this without further comment.) We denote by $K_p(G)$ the unique smallest normal subgroup of G for which the quotient $G/K_p(G)$ is a p-group. $G/K_p(G)$ is called the *p-residual* of G. W. Gaschütz (**2**, Satz 7) has proved the following

Theorem. *Set* $K = K_p(G)$. *If the Sylow p-subgroups of* K *are abelian, then* G *splits over* K.

The method of proof adopted in (**2**) depends on general reduction theorems for the splitting of a group over an abelian normal subgroup, and these require quite elaborate calculations with factor systems. The techniques involved are powerful enough to yield many other interesting results, but the Theorem quoted above is perhaps of sufficient intrinsic interest to warrant the publication of another proof. This is the purpose of the present note. The chief tools employed here are the Second Theorem of Grün (**7**, p. 171, Theorem 6), the Theorems of Schur and Zassenhaus (**7**, p. 162, Theorems 25 and 27), and

Lemma 1. *If the group* H *has an abelian Sylow p-subgroup* P, *then* $H' \cap Z(H) \cap P = 1$ (*where* H' *is the derived group of* H, *and* $Z(H)$ *is the centre of* H).

This was proved for soluble H by D. R. Taunt (**5**), and in general by means of an easy transfer argument by B. Huppert (**4**).

2. Given the results mentioned above, the key to the proof of the Theorem lies in certain observations about p-complements. It is convenient to begin by introducing some notation and terminology. Suppose that H is a group, and K a subgroup of H. We write K^H for the *normal closure* of K in H, that is the unique smallest normal subgroup of H which contains K. We write $N_H(K)$ for the *normalizer* of K in H, that is the unique largest subgroup of H in which K is contained as a normal subgroup. Thus K is normal in H if and only if $K^H = K$, and if and only if $N_H(K) = H$. We say that K is *contranormal* † in H if $K^H = H$; and that K is *self-normalizing* in H if $N_H(K) = K$. In general, K may be contranormal but not self-normalizing in H, or self-normalizing but not contranormal in H; but we shall be concerned with special circumstances in which each of these properties implies the other. We may remark that the *abnormal* subgroups of R. W. Carter (see for instance (**1**)), of which normalizers of Sylow subgroups provide the most familiar example, are both contranormal and self-normalizing.

† This term has been introduced by Professor P. Hall.

A subgroup Q is called a *p-complement* of a group H if Q has order prime to p, and index in H a power of p. In general a group need not possess a p-complement. However, we note the obvious fact that if a group H has a p-complement Q, then $Q^H = K_p(H)$. In particular, Q is contranormal in H if and only if $K_p(H) = H$, that is if and only if H has trivial p-residual. A group with trivial p-residual is said to be *p-perfect*.

By Schur's Theorem, a sufficient condition for the existence of a p-complement in a group H is that H has a normal Sylow p-subgroup P, for then H splits over P; and in that case Zassenhaus's Theorem shows that the p-complements form a single conjugacy class of subgroups in H. We deduce

Lemma 2. *If H is a normal subgroup of the group G, and H has a normal Sylow p-subgroup, then $G = HN_G(Q)$, where Q is any p-complement of H.*

Proof. This goes by the usual Frattini argument. For any $g \in G$, we have $g^{-1}Qg \leqq H$; and then, since $g^{-1}Qg$ is a p-complement of H, we know that there exists $h \in H$ such that $g^{-1}Qg = h^{-1}Qh$. Then $gh^{-1} \in N_G(Q)$, and the result follows.

Now suppose that the group H has a normal Sylow p-subgroup, and let Q be a p-complement of H. It follows from the fact that the p-complements form a single conjugacy class in H that $N_H(Q)$ is abnormal in H—by an argument exactly similar to the one used in proving that the normalizer of a Sylow subgroup is abnormal in a group. Hence if Q is self-normalizing in H, then Q is abnormal in H, and so in particular Q is contranormal in H. The converse is false: a group H may have a normal Sylow p-subgroup, and a p-complement Q which is contranormal but not self-normalizing in H. For instance, in a split extension of a quaternion group by a cyclic group of order 3, defined by means of an automorphism permuting cyclically the three subgroups of order 4 in the quaternion group, a 2-complement is contranormal, but is of index 2 in its normalizer. However, we shall see that the imposition of an extra condition makes this converse true.

Lemma 3. *Suppose that the group H has a normal abelian Sylow p-subgroup P, and let Q be a p-complement of H. If Q is contranormal in H, then Q is self-normalizing in H.*

Proof. Let $P_0 = P \cap N_H(Q)$. Since P is normal in H, P_0 is normal in $N_H(Q)$. Since also Q is normal in $N_H(Q)$ and $P_0 \cap Q = 1$, $N_H(Q) = P_0 \times Q$ (direct product), and P_0 centralizes Q. Moreover, P_0 centralizes P, since P is abelian. Therefore P_0 centralizes $PQ = H$, that is $P_0 \leqq Z(H)$, the centre of H.

Now by hypothesis $Q^H = H$, or equivalently, H is p-perfect. It follows from this that $P \leqq H'$, the derived group of H. Hence $P_0 \leqq H' \cap Z(H) \cap P = 1$, by Lemma 1. Therefore $N_H(Q) = Q$.

3. Proof of the Theorem

We proceed by induction on the group order. Let P be a Sylow p-subgroup of K. P is abelian, so that K is p-normal in the sense of Grün. By definition of K, K is p-perfect, and therefore the Second Theorem of Grün shows that $N_K(P)$ is p-perfect. Let $N = N_G(P)$; then $N \cap K = N_K(P)$. $N \cap K$ is normal in N, and since $N/N \cap K$ is isomorphic to NK/K, $N/N \cap K$ is a p-group. Since we have shown that $N \cap K$ is p-perfect, we must have $N \cap K = K_p(N)$. As a subgroup of K, $N \cap K$ has abelian Sylow p-subgroups. Hence if N is a proper subgroup of G, the induction hypothesis implies that N splits over $N \cap K$, say $N = (N \cap K)P^*$ with $(N \cap K) \cap P^* = 1$. Then, since $P^* \leqq N$, $K \cap P^* = 1$. Therefore, since the Frattini argument shows that $G = KN$, we have $G = KP^*$ with $K \cap P^* = 1$. Thus G splits over K.

So we may suppose now that $N = G$, that is that P is normal in G. Then P is normal in K, so that by Schur's Theorem, K possesses a p-complement Q. K is p-perfect, and so $Q^K = K$. Since P is abelian, it follows from Lemma 3 that $N_K(Q) = Q$.

Now by Lemma 2, $G = KN_G(Q)$. Let P^* be a Sylow p-subgroup of $N_G(Q)$, so that $N_G(Q) = QP^*$. Then, since $Q \leqq K$, $G = KP^*$. Also

$$Q = N_K(Q) = K \cap N_G(Q) = K \cap (QP^*) = Q(K \cap P^*),$$

so that as $K \cap P^*$ is a p-group contained in Q, which has order prime to p, we must have $K \cap P^* = 1$. Thus G splits over K. This completes the induction argument.

4. P. Hall (**3**, § 5) has described examples of groups which show that the condition that the Sylow p-subgroups of K be abelian is indispensable for the truth of the Theorem. For instance, let X be a non-abelian group of order 27 and exponent 3. The centre Z of X has order 3. The automorphism group of X has a quaternion subgroup Q which leaves Z invariant. If we form a split extension H of X by means of Q, and then extend H by an element a commuting with every element of H and such that a^3 generates Z, we obtain a group G of order 648. $K_3(G) = H$, but H contains all elements of G of order 3, so that G cannot split over H.

5. In this final section, we add some remarks on a particular application of the Theorem. A group G is said to be *p-nilpotent* if $K_p(G)$ has order prime to p (in which case $K_p(G)$ is the unique p-complement in G). Thus the extremes of behaviour for a group, in regard to its p-residual, are to be p-nilpotent or p-perfect. The Theorem has the following

Corollary. *If G is a group with cyclic Sylow p-subgroups, then either G is p-nilpotent or G is p-perfect.*

Proof. Let $K = K_p(G)$. The hypothesis for G is inherited by subgroups, so that the Sylow p-subgroups of K are cyclic. Hence by the Theorem, G splits over K, and so there is a p-subgroup P of G with $G = KP$ and $K \cap P = 1$. Let $\bar{P}$ be a Sylow p-subgroup of G containing P. $K \cap \bar{P}$ is normal in $\bar{P}$, and since $G = KP$, $\bar{P} = (K \cap \bar{P})P$; and of course $(K \cap \bar{P}) \cap P = 1$. Since $\bar{P}$ is cyclic, it follows that either $P = \bar{P}$ or $P = 1$. This gives the result.

Further properties of groups with cyclic Sylow p-subgroups have been obtained by H. Wielandt (**6**, § 4).

We may note that if $p = 2$ in this Corollary, then G is actually 2-nilpotent. This may be proved by an elementary argument, as follows.

Proposition. *If G is a group with cyclic Sylow 2-subgroups, then G is 2-nilpotent.*

Proof. Let ρ be the right regular representation of G. Suppose that G has order $2^m n$, where n is odd and we may assume that $m>0$. Let x be a generator of a Sylow 2-subgroup of G. Then $\rho(x)$ may be expressed as a product of n disjoint cycles of equal length 2^m, and hence $\rho(x)$ is an odd permutation. Therefore the even permutations form a subgroup of index 2 in $\rho(G)$. By the isomorphism between G and $\rho(G)$, G has a subgroup G_1 of index 2. If G_1 has even order, G_1 is again a group with cyclic Sylow 2-subgroups, so that we may repeat the argument above to show that G_1 has a subgroup G_2 of index 2. Repetition of the argument yields a chain of subgroups

$$G = G_0 > G_1 > \ldots > G_m,$$

with G_i of index 2 in G_{i-1} for each $i = 1, \ldots, m$. G_m has odd order. By induction on m, we show that G_m is normal in G, so that $G_m = K_2(G)$ and G is 2-nilpotent.

Remarks. 1^0. Since groups of odd order are soluble (Feit-Thompson Theorem), we may deduce from this result that a group with cyclic Sylow 2-subgroups is necessarily soluble.

2^0. If, in place of the elementary argument used above, we apply the Theorem of Burnside which asserts that if a group G has a Sylow p-subgroup in the centre of its normalizer, then G is p-nilpotent, we can obtain the following stronger result. If G is a group with cyclic Sylow p-subgroups, where p is the least prime factor of the order of G, then G is p-nilpotent. This is known.

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FINITE GROUPS WITH PRESCRIBED SYLOW TOWER SUBGROUPS

By JOHN S. ROSE

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FINITE GROUPS WITH PRESCRIBED SYLOW TOWER SUBGROUPS

By JOHN S. ROSE

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B. Huppert ((**5**) Satz 22) proved that if all the proper subgroups of a finite group G are supersoluble then G is soluble; and that if in addition $|G|$ has at least 4 distinct prime factors then G is itself supersoluble. More recently, in (**10**), the effects of replacing *proper* by *proper abnormal* in the hypothesis of this result have been investigated, and the following facts established.

(i) *There exists a finite insoluble group, all of whose proper abnormal subgroups are supersoluble.*

However, Huppert's theorem extends to

(ii) *If all the proper self-normalizing subgroups of a finite group G are supersoluble, then G is soluble.*

There is also a partial extension:

(iii) *If all the abnormal maximal subgroups of a finite group G are supersoluble and have prime-power indices in G, then G is soluble.*

The present paper furnishes another partial extension of Huppert's theorem (in the corollary to Theorem 1); more detailed information on the structure of soluble but non-supersoluble groups, all of whose proper abnormal subgroups are supersoluble; and various related results. The terminology and notation of (**10**) are adopted here. All groups considered are finite.

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Attention is directed largely to the existence of Sylow towers. A group G is called a *Sylow tower group* when every non-trivial homomorphic image of G has a non-trivial normal Sylow subgroup. This is true if and only if, for some ordering of distinct prime numbers $p_1, p_2, \ldots, p_n$, there exists a series of normal subgroups of G:

$$1 = G_0 \leqslant G_1 \leqslant \ldots \leqslant G_n = G$$

such that $G_i/G_{i-1} \cong$ a Sylow p_i-subgroup of G $(i = 1, 2, \ldots, n)$. Such a series will be called a *Sylow tower of G of complexion $p_1, p_2, \ldots, p_n$*. G_i/G_{i-1} is permitted to be trivial: this happens of course if and only if

p_i does not divide $|G|$. In any case, if G has a Sylow tower of complexion $p_1, \ldots, p_n$, all the prime factors of $|G|$ appear among $p_1, \ldots, p_n$. It is clear that if G is a group with a Sylow tower of complexion $p_1, \ldots, p_n$, then subgroups and quotients of G also have Sylow towers of the same complexion.

In particular, if $p_1 > p_2 > \ldots > p_n$, G is said to have an *ordered* Sylow tower. One of the most striking properties of supersoluble groups is that they possess ordered Sylow towers: see, for instance, Huppert ((**5**) Satz 7).

THEOREM 1. *Suppose that every proper abnormal subgroup of the finite group G has a Sylow tower of complexion $p_1, \ldots, p_n$; and that the Sylow 2-subgroups of G are abelian. Then G is soluble.*

Proof. Suppose the result false, and let the group G provide a counterexample of least possible order. Since the properties of G, as enunciated in the statement of the theorem, are inherited by quotients of G, the minimality of G implies that G has no non-trivial soluble normal subgroup. Every prime factor p of $|G|$ appears among $p_1, \ldots, p_n$; for if P is a Sylow p-subgroup of G then P is not normal in G, and so $N_G(P)$ is a proper abnormal subgroup of G; therefore $N_G(P)$ has by hypothesis a Sylow tower of complexion $p_1, \ldots, p_n$, and p certainly divides $|N_G(P)|$.

A positive integer r ($\leqslant n$) is uniquely defined by the following conditions: G has a series of normal subgroups $G_r \leqslant G_{r+1} \leqslant \ldots \leqslant G_n = G$, such that G_r is not p_r-nilpotent and (if $r < n$) $G_i/G_{i-1} \cong$ a Sylow p_i-subgroup of G for $i = r+1, \ldots, n$. Let $H = G_r$. We distinguish two possibilities:

(a) $p_r = 2$. Then if T is a Sylow 2-subgroup of H (and so also of G), $N_G(T)$ is a proper abnormal subgroup of G, and therefore has a Sylow tower of complexion $p_1, \ldots, p_n$; hence so also has $H \cap N_G(T) = N_H(T)$. Since $|H|$ is not divisible by $p_{r+1}, \ldots, p_n$, this means that $N_H(T)$ has a Sylow tower of complexion $p_1, \ldots, p_r$. In particular, $N_H(T)$ is p_r-nilpotent, that is 2-nilpotent. But since by hypothesis T is abelian, this implies that T lies in the centre of $N_H(T)$, and therefore, by a well-known theorem of Burnside ((**2**) 327), that H is 2-nilpotent; that is G_r is p_r-nilpotent. This is in contradiction to the definition of r.

(b) p_r is odd. Let P be a Sylow p_r-subgroup of H (and so of G), and let P_0 be any non-trivial characteristic subgroup of P. Then $N_G(P_0) \geqslant N_G(P)$, and so $N_G(P_0)$ is a proper abnormal subgroup of G. $N_G(P_0)$ has a Sylow tower of complexion $p_1, \ldots, p_n$; hence so also has $N_H(P_0)$. Since $|H|$ is not divisible by $p_{r+1}, \ldots, p_n$, $N_H(P_0)$ has a Sylow tower of complexion $p_1, \ldots, p_r$. In particular, $N_H(P_0)$ is p_r-nilpotent.

This implies that $N_H(P_0)/C_H(P_0)$ is a p_r-group. A theorem of J. G. Thompson (**12**) now shows that H is p_r-nilpotent, which again contradicts the definition of r.

In each case we are led to a contradiction, and therefore conclude that the result as stated is true.

As an immediate deduction we have the

COROLLARY. *If every proper abnormal subgroup of the finite group G is supersoluble, and if the Sylow 2-subgroups of G are abelian, then G is soluble.*

It is of course essential for the validity of Theorem 1 that the proper abnormal subgroups have Sylow towers of the same complexion, since, for example, in the icosahedral group every proper subgroup has a Sylow tower and all the Sylow subgroups are abelian.

We shall now seek to discover how nearly a group satisfying the hypotheses of Theorem 1 itself possesses a Sylow tower of complexion $p_1, \ldots, p_n$. With this purpose in view, we consider first soluble groups in which all the proper abnormal subgroups have normal Sylow p-subgroups, for a fixed prime p. If a group G has a normal Sylow p-subgroup, then so also does every proper subgroup of G; but the converse is false. However, we have

THEOREM 2. *Suppose that K is a finite soluble group in which every proper abnormal subgroup has a normal Sylow p-subgroup, but that K does not have a normal Sylow p-subgroup. Then the following conclusions hold:*

(i) *Let P be a Sylow p-subgroup of K. $N_K(P)$ is a maximal subgroup of K, so that $|K:N_K(P)|$ is a power of a prime, say q.*

(ii) *Any abnormal maximal subgroup L of K which is not conjugate to $N_K(P)$ in K has index in K a power of p, and the unique Sylow p-subgroup of L is precisely the largest normal p-subgroup of K.*

(iii) *K has at most two conjugacy classes of abnormal maximal subgroups.*

(iv) *If Q is the largest normal q-subgroup of K, then $Q \nleq N_K(P)$ and therefore $PQ \lhd K$. Moreover, K/PQ is nilpotent.*

In particular, K has nilpotent length $\leqslant 3$, and t-length 1 for every prime factor t of $|K|$ except possibly for $t = q$; and in any case K has q-length $\leqslant 2$.

In proving Theorem 2, we shall use the following lemma, which is a slight extension of a well-known result of W. Gaschütz ((**4**) Satz 10), and which follows from this and another result of Gaschütz.

LEMMA 1. *Let G be any non-nilpotent finite group, and let Γ be the intersection of all abnormal maximal subgroups of G. Suppose that $H \lhd G$*

and that $H \leqslant \Gamma$. Then, if $K \lhd G$, $H \leqslant K$, and K/H is nilpotent, K is nilpotent.

Proof. Let Φ be the Frattini subgroup of G. Then $\Phi \leqslant \Gamma$, and Γ/Φ is the centre of G/Φ ((**4**) Satz 15). Let $\bar{H} = H\Phi$ and $\bar{K} = K\Phi$. $\bar{K}/\bar{H}$ is isomorphic to a quotient of K/H, so that $\bar{K}/\bar{H}$ is nilpotent. Therefore, since $\Phi \leqslant \bar{H} \leqslant \Gamma$ and hence $\bar{H}/\Phi \leqslant$ the centre of G/Φ, $\bar{K}/\Phi$ is nilpotent. It follows that $\bar{K}$ and K are nilpotent ((**4**) Satz 10).

Proof of Theorem 2. By hypothesis, $N_K(P) < K$. If M is a maximal subgroup of K such that $M \geqslant N_K(P)$, then M is abnormal in K and so, by hypothesis, $P \lhd M$; hence $M = N_K(P)$. By a well-known theorem of Galois for soluble groups, $|K:M|$ is a power of a prime, say q.

(a) If K has a single conjugacy class of abnormal maximal subgroups, namely the normalizers of the Sylow p-subgroups, then K has a normal Sylow q-subgroup Q, and K/Q is nilpotent: see ((**10**) proof of Theorem 1). In this case, Theorem 2 is proved.

(b) Suppose, now, that K has more than one conjugacy class of abnormal maximal subgroups. Let L be an abnormal maximal subgroup of K not conjugate to M. By hypothesis, L has a normal Sylow p-subgroup, P_1 say. P_1 lies in some Sylow p-subgroup of K; we may suppose that $P_1 \leqslant P$. Since L is not the normalizer of a Sylow p-subgroup of K, $P_1 < P$. It follows from this, since $|K:L|$ is a power of a prime, that $|K:L|$ is a power of p. It also follows from $P_1 < P$ that $P_1 < N_P(P_1)$. Since $P_1 \lhd L$, L is maximal in K, and P_1 is the Sylow p-subgroup of L, this implies that $P_1 \lhd K$.

Let P_0 be the largest normal p-subgroup of K. Then $P_1 \leqslant P_0$, and we want to show that $P_1 = P_0$. If R is a p-complement of L then $L = P_1R$ and $L \leqslant P_0R < K$, since P_0 is not a Sylow p-subgroup of K. The maximality of L now implies that $L = P_0R$, and hence that $P_1 = P_0$.

Now let Γ be the intersection of all abnormal maximal subgroups of K. We have shown that $P_0 \leqslant \Gamma$, and so the number of conjugacy classes of abnormal maximal subgroups of K is equal to the corresponding number for K/P_0. Any abnormal maximal subgroup L/P_0 of K/P_0, not conjugate to M/P_0 in K/P_0, has index a power of p, and Sylow p-subgroup P_0/P_0; that is, L/P_0 is a p-complement of K/P_0. Since the p-complements form a single conjugacy class of subgroups of K/P_0, we conclude that K has just two conjugacy classes of abnormal maximal subgroups.

We want to show that $Q \nleqslant M$, where Q is the largest normal q-subgroup of K. Suppose to the contrary that $M \geqslant Q$, and therefore that every conjugate of M in K contains Q. Since every subgroup of index a power of p in K certainly contains Q, $Q \leqslant \Gamma$. Thus $P_0Q \leqslant \Gamma$. If V/P_0Q is the largest normal p-subgroup of K/P_0Q, Lemma 1 implies that V is nilpotent. If V_p is the Sylow p-subgroup of V then $V_p \lhd K$ and so $V_p = P_0$. Hence K/P_0Q has no non-trivial normal p-subgroup. Exactly similar reasoning shows that K/P_0Q has no non-trivial normal q-subgroup. If t is any prime factor of $|K|$ with $t \neq p$ and $t \neq q$, and if R_t is a t-complement of K, then $R_t \lhd K$; for $N_K(R_t)$ is abnormal in K, but no proper abnormal subgroup of K has index a power of t. Hence K has a normal Hall $\{p,q\}$-subgroup H, and K/H is nilpotent. Certainly $P_0Q \leqslant H$, and in view of the deduction above that K/P_0Q has no non-trivial normal p-subgroup and no non-trivial normal q-subgroup, we must conclude that $P_0Q = H$. But then $H \leqslant \Gamma$ and K/H is nilpotent, so that it follows by Lemma 1 that K is nilpotent. This however contradicts the hypothesis that K does not have a normal Sylow p-subgroup.

Hence $Q \nleqslant M$. Therefore $MQ = K$, and so it follows from $P \lhd M$ that $PQ \lhd K$. No proper subgroup of K of index a power of p can contain PQ, and since also $M \ngeqslant Q$, no abnormal maximal subgroup of K contains PQ. Therefore K/PQ is nilpotent.

Theorem 2 will now be applied to the consideration of soluble groups in which all the proper abnormal subgroups have Sylow towers of complexion $p_1, \ldots, p_n$.

THEOREM 3. *Suppose that G is a finite soluble group in which every proper abnormal subgroup has a Sylow tower of complexion $p_1, \ldots, p_n$, but that G does not have a Sylow tower of the same complexion. Then*

(a) *There is at most one prime factor q of $|G|$ which does not appear among $p_1, \ldots, p_n$. If there is such a q, then the q-complements of G are the only proper abnormal subgroups of G, G has a normal Sylow q-subgroup Q, and G/Q is nilpotent.*

(b) *If all the prime factors of $|G|$ appear among $p_1, \ldots, p_n$, then there is an integer r satisfying $0 \leqslant r < n-1$, and a direct decomposition of G: $G = H \times K$, such that if $r = 0$ then $H = 1$, while if $r > 0$ then H is the unique Hall $\{p_1, \ldots, p_r\}$-subgroup of G and H is nilpotent; and K satisfies Theorem 2, with $p = p_{r+1}$.*

In particular, G has nilpotent length $\leqslant 3$, and t-length 1 for every prime factor t of $|G|$ with one possible exception; and in any case G has t-length $\leqslant 2$.

Proof. (a) Suppose first that there is a prime factor q of $|G|$ which does not appear among $p_1, \ldots, p_n$. Then if M is any abnormal maximal subgroup of G, q does not divide $|M|$ and therefore, since G is soluble, M must be a q-complement of G. Thus q is unique and G has a single conjugacy class of abnormal maximal subgroups, namely its q-complements. This implies (see ((**10**) proof of Theorem 1)) that G has a normal Sylow

q-subgroup Q, and that G/Q is nilpotent. Therefore the q-complements of G are nilpotent. Every proper abnormal subgroup of G lies in an abnormal maximal subgroup of G, that is in a q-complement of G. Since the q-complements are nilpotent, it follows that they are the only proper abnormal subgroups of G.

(b) Now suppose that all the prime factors of $|G|$ appear among $p_1, \ldots, p_n$. An integer r, such that $0 \leqslant r < n-1$, is uniquely defined by the following conditions: G has a series of normal subgroups

$$1 = G_0 \leqslant G_1 \leqslant \ldots \leqslant G_r,$$

such that G/G_r does not have a normal Sylow p_{r+1}-subgroup, and (if $r > 0$) $G_i/G_{i-1} \cong$ a Sylow p_i-subgroup of G for $i = 1, \ldots, r$. Let $H = G_r$. Then if $r = 0$, $H = 1$, while if $r > 0$, H is the unique Hall $\{p_1, \ldots, p_r\}$-subgroup of G. Let K be a Hall $\{p_{r+1}, \ldots, p_n\}$-subgroup of G. If K were contained in any proper abnormal subgroup of G, then K would have a Sylow tower of complexion $p_1, \ldots, p_n$; and since K is a $\{p_{r+1}, \ldots, p_n\}$-group, this would mean that K had a Sylow tower of complexion $p_{r+1}, \ldots, p_n$. But $K \cong G/H$, and by definition of r and H, G/H does not have a normal Sylow p_{r+1}-subgroup. Hence K is contained in no proper abnormal subgroup of G. Since K is a Hall subgroup of the soluble group G, $N_G(K)$ is abnormal in G. Therefore $K \lhd G$, and G/K is nilpotent. It follows that $G = H \times K$, and then, since $H \cong G/K$, that H is nilpotent. K ($\cong G/H$) is a group in which every proper abnormal subgroup has a normal Sylow p_{r+1}-subgroup, but K itself does not have a normal Sylow p_{r+1}-subgroup. This completes the proof.

We consider next soluble groups in which all the proper abnormal subgroups are supersoluble.

THEOREM 4. *Suppose that G is a finite soluble but not supersoluble group, in which every proper abnormal subgroup is supersoluble. Let the distinct prime factors of $|G|$ be $p_1 > p_2 > \ldots > p_n$. Then there exist an integer r satisfying $0 \leqslant r < n-1$, and a direct decomposition of G: $G = H \times K$, such that if $r = 0$ then $H = 1$, while if $r > 0$ then H is the unique Hall $\{p_1, \ldots, p_r\}$-subgroup of G and H is nilpotent; and either K has a normal Sylow p_{r+1}-subgroup, L say, and G/L is supersoluble (so that in particular G has an ordered Sylow tower), or K satisfies Theorem 2, with $p = p_{r+1}$.*

Proof. Every proper abnormal subgroup of G has a Sylow tower of complexion $p_1, \ldots, p_n$, so that if G does not itself have an ordered Sylow tower, Theorem 3(b) yields the result at once.

Assume now that G possesses an ordered Sylow tower:

$$1 = G_0 < G_1 < \ldots < G_n = G,$$

with each $G_i \lhd G$ and $G_i/G_{i-1} \cong$ a Sylow p_i-subgroup of G. For each $i = 0, 1, \ldots, n-1$, let K_i be a Hall $\{p_{i+1}, \ldots, p_n\}$-subgroup of G, and let $K_n = 1$. Thus, for each i, K_i complements G_i in G. The K_i may be chosen so that $G = K_0 > K_1 > \ldots > K_n = 1$. Let $r \geqslant 0$ be the largest integer such that G_r is nilpotent and $K_r \lhd G$. Then $G = G_r \times K_r$, and certainly $r < n-1$. Either G_{r+1} is not nilpotent or $K_{r+1} \ntriangleleft G$. If $K_{r+1} \ntriangleleft G$ then $N_G(K_{r+1})$ is a proper abnormal subgroup of G, and so K_{r+1} is supersoluble. On the other hand, if $K_{r+1} \lhd G$ then G_{r+1} is not nilpotent and so, since $G/K_{r+1} \cong G_{r+1}$, K_{r+1} lies in some proper abnormal subgroup of G, and hence K_{r+1} is supersoluble. Thus, in any event, K_{r+1} is supersoluble. Since $K_r \cong G/G_r$, K_r has a normal Sylow p_{r+1}-subgroup, say L; and $K_r/L \cong K_{r+1}$, so that K_r/L is supersoluble. It follows that $L \lhd G$ and, since G_r is nilpotent, that G/L is supersoluble. The result follows by setting $H = G_r$ and $K = K_r$.

It may be observed, as a corollary of Theorem 2, that if K is a soluble group in which every proper abnormal subgroup has a normal Sylow p-subgroup, then K has a normal subgroup Q, of prime-power order, such that K/Q has a normal Sylow p-subgroup. Theorems 3 and 4 yield corresponding statements. These are particular cases of a rather general result, which may be deduced very easily from Lemma 1.

A class $\mathfrak{X}$ of groups is called Q-*closed* when any homomorphic image of an $\mathfrak{X}$-group is again an $\mathfrak{X}$-group.

THEOREM 5. *Let $\mathfrak{X}$ be any* Q-*closed class of groups. Suppose that G is a finite soluble but non-nilpotent group in which every abnormal maximal subgroup is an $\mathfrak{X}$-group. Then G has a normal subgroup W of prime-power order such that G/W is an $\mathfrak{X}$-group.*

Proof. Let U be the Fitting subgroup of G, and let Γ be the intersection of all abnormal maximal subgroups of G. Γ is nilpotent (Gaschütz ((**4**) Satz 16)) so that $\Gamma \leqslant U$. G/U is non-trivial, and since it is also soluble there is a non-trivial nilpotent normal subgroup N/U of G/U. This implies that $\Gamma < U$, for otherwise, by Lemma 1, N would be a nilpotent normal subgroup of G with $N > U$, contrary to the definition of U. Hence there is an abnormal maximal subgroup M of G such that $M \ngeqslant U$. There must be at least one Sylow subgroup W of U such that $M \ngeqslant W$. Then $W \lhd G$ and $MW = G$. $G/W \cong M/M \cap W \in \mathfrak{X}$, since $M \in \mathfrak{X}$ and $\mathfrak{X}$ is Q-closed.

Remark. The substance of this argument was used in proving the Corollary to Theorem 4 in (**10**). I am indebted to Dr R. W. Carter for pointing out to me that it is more generally applicable.

It seems desirable now to investigate the existence of groups satisfying Theorems 2, 3, and 4. We shall confine attention to such groups G having q-length 2 for some prime factor q of $|G|$, and therefore having also nilpotent length 3. Existence is shown already by the symmetric group of degree 4, which has 2-length 2, and in which every proper abnormal subgroup is supersoluble. But, in view of the second assertion of Huppert's theorem, it is of interest to inquire whether there is a bound on the number of distinct prime factors of $|G|$, at least when all the proper abnormal subgroups of G are supersoluble. It will be shown that there is no such bound, even when G has no non-trivial decomposition as a direct product. First, however, we shall show that there are certain arithmetical restrictions when all the proper abnormal subgroups of G are supersoluble.

We begin by observing that if G_1 is any finite nilpotent group and G_2 is a finite soluble group in which every proper abnormal subgroup is supersoluble, then $G_1 \times G_2$ is also a soluble group in which every proper abnormal subgroup is supersoluble. In order to prove this, let M be an abnormal maximal subgroup of $G_1 \times G_2$. By a result of N. Itô ((**6**) Proposition 14), either $M \geqslant G_1$ or $M \geqslant G_2$. Since $G_1 \times G_2/G_2 \simeq G_1$, which is by hypothesis nilpotent, $M \ngeqslant G_2$. Therefore $M \geqslant G_1$, and so $M = G_1 \times (M \cap G_2)$. By the isomorphism $G_1 \times G_2/G_1 \simeq G_2$, $M \cap G_2$ is abnormal maximal in G_2, so that by hypothesis $M \cap G_2$ is supersoluble. Hence M is supersoluble.

Thus the situation of interest is that described in the hypotheses of

THEOREM 6. *Suppose that G is a finite soluble group in which every proper abnormal subgroup is supersoluble, that G has q-length 2 for some prime factor q of $|G|$, and that G has no non-trivial Sylow subgroup as a direct factor. Then, if p is the largest prime factor of $|G|$, $p \equiv 1$ (q); and if t is any prime factor of $|G|$ other than p or q, $q \equiv 1$ (t).*

Proof. We apply Theorem 4 to G. Since G has no non-trivial Sylow subgroup as a direct factor, $r = 0$, $H = 1$, and $G = K$. Since G has q-length 2, G does not have a Sylow tower, and so K satisfies Theorem 2, with p the largest prime factor of $|G|$. We now adopt the notation of Theorem 2. We see that q, the prime factor of $|K|$ for which K has q-length 2, is also the prime of which $|K : N_K(P)|$ is a power, where P is a Sylow p-subgroup of K. Let $M = N_K(P)$. By Theorem 2(iv), $M \ngeqslant Q$, where Q is the largest normal q-subgroup of K. $M \cap Q \lhd M$, and since $M \cap Q < Q$ and Q has prime-power order, $M \cap Q < N_Q(M \cap Q)$. Therefore $M \cap Q \lhd K$. Also $M \cap Q \leqslant \Gamma$, the intersection of all abnormal maximal subgroups of K, since $M \cap Q \leqslant M$ and every abnormal maximal subgroup of K not conjugate to M has index in K a power of p.

Since K has q-length 2, there exists an abnormal maximal subgroup L of K not conjugate to M in K. Since $MQ = K$ and, by hypothesis, M is supersoluble, K/Q is supersoluble. Therefore every maximal subgroup of K/Q has prime index in K/Q. It follows, by Theorem 2(ii), that $|K:L| = p$ and that $|P:P_0| = p$, where P_0 is the largest normal p-subgroup of K. Since $P_0 \leqslant L \cap M$, Theorem 2(iii) shows also that $P_0 \leqslant \Gamma$.

We shall now show that if t is any prime factor of $|\Gamma|$ then t also divides $|K/\Gamma|$. Let T be the Sylow t-subgroup of the nilpotent group Γ. Then $T \lhd K$. If K/T were supersoluble then, by Lemma 1, K would have nilpotent length 2, in contradiction to the hypothesis that K has q-length 2. Therefore K/T is not supersoluble. If t did not divide $|K/\Gamma|$ then T would be the unique Sylow t-subgroup of K. A t-complement R of K would be isomorphic to K/T, and therefore not supersoluble. Since $N_K(R)$ is abnormal in K, this would imply that $N_K(R) = K$, that is $R \lhd K$. But then K would have the decomposition $K = T \times R$, in contradiction to the hypothesis that K has no non-trivial Sylow subgroup as a direct factor.

We note that $Q\Gamma$ is the Fitting subgroup of K, since $P_0 \leqslant \Gamma$ and, by Theorem 2, any normal $\{p, q\}'$-subgroup of K lies in Γ. Hence, by Lemma 1, $Q\Gamma/\Gamma$ is the Fitting subgroup of K/Γ. It follows that K/Γ has q-length 2.

These facts enable us, without loss of generality in proving Theorem 6, to assume that $\Gamma = 1$. Then $P_0 = 1$, and so $|P| = p$. L is a p-complement of K. Let $\bar{K} = K/Q$, and let $\bar{P} = PQ/Q$. $\bar{P} \lhd \bar{K}$, by Theorem 2(iv), and $|\bar{P}| = p$. Let $\bar{H} = C_{\bar{K}}(\bar{P})$. $\bar{K}/\bar{H}$ is cyclic, of order dividing $p-1$. Let $\bar{L} = L/Q$, so that $\bar{K} = \bar{P}\bar{L}$ and, by Theorem 2(iv), $\bar{L}$ is nilpotent. Then $\bar{H} = \bar{P}.C_{\bar{L}}(\bar{P}) = \bar{P} \times C_{\bar{L}}(\bar{P})$. Thus $\bar{H}$ is nilpotent. Hence if U/Q is the Fitting subgroup of K/Q then K/U is cyclic, of order dividing $p-1$. In particular, since K has q-length 2, q must be a divisor of $|K/U|$. Hence $p \equiv 1\ (q)$.

Next, since $M \cap Q \leqslant \Gamma$, $M \cap Q = 1$. Since M is a maximal subgroup of K, it follows that Q is a minimal normal subgroup of K. In fact Q is the unique minimal normal subgroup of K, since it is also the Fitting subgroup of K. Let V/Q be a q-complement of L/Q. V is supersoluble, since L is by hypothesis supersoluble. The elementary abelian q-group Q has the structure of a representation space for V/Q over the Galois field $\mathrm{GF}(q)$, in a natural way. Since q does not divide $|V/Q|$, Maschke's theorem implies the complete reducibility of the representation space. This means that Q is expressible as a direct product of minimal normal subgroups of V, say $Q = Q_1 \times \ldots \times Q_s$, where each Q_i is a minimal normal subgroup of V $(i = 1, \dots, s)$. It follows from the fact that V is supersoluble that each $|Q_i| = q$ $(i = 1, \dots, s)$. Therefore $V/C_V(Q_i)$ is a subgroup of a cyclic group of order $q-1$ $(i = 1, \dots, s)$. Since Q is the Fitting subgroup of the soluble group K, $C_K(Q) = Q$ (H. Fitting ((**3**) 106, Hilfssatz 12)). Hence $\bigcap_{i=1}^{s} C_V(Q_i) = C_V(Q) = Q$. It follows then that V/Q is isomorphic to a subgroup of the direct product of s copies of a cyclic group of order $q-1$. However, V/Q is also isomorphic to a Hall $\{p,q\}'$-subgroup of K. Therefore if t is any prime factor of $|K|$ such that $t \neq p$ and $t \neq q$, then t divides $(q-1)^s$, and so $q \equiv 1$ (t). This establishes Theorem 6.

THEOREM 7. *For any integer $n \geqslant 2$, there exists a group G satisfying Theorem* 6, *and such that $|G|$ has n distinct prime factors.*

We make use of

LEMMA 2. *Let q be any prime number, and suppose that the finite group H is an extension of a q-group Q by an abelian group X. If every element x of X satisfies the equation $x^{q-1} = 1$, then H is supersoluble.*

Proof. Let K/L be any chief factor of H with $K \leqslant Q$. To establish the lemma, it suffices to show that K/L is cyclic. K/L has the structure of an irreducible A-module, where A is the group algebra over $\mathrm{GF}(q)$ of the group $Y = H/C_H(K/L)$. Now $C_H(K/L) \geqslant Q$ (R. Baer ((**1**) 649, Proposition 1)), so that Y is isomorphic to a quotient group of X. Thus Lemma 2 follows from

LEMMA 3. *Suppose that M is an irreducible A-module, where A is the group algebra over* $\mathrm{GF}(q)$ *of a finite abelian group X. If every element x of X satisfies the equation $x^{q-1} = 1$, then M has dimension* 1 *over* $\mathrm{GF}(q)$.

Proof. Set $F = \mathrm{GF}(q)$. Let $\bar{X}$ be the group of F-automorphisms of the F-space M induced by X, and $\bar{A}$ the algebra of F-endomorphisms of M induced by A. For any m in M, a in A, and a' in A, $(ma')a = (ma)a'$, since A is commutative. Therefore $\bar{A}$ is a subalgebra of the algebra of A-endomorphisms of M, which by Schur's lemma is a division algebra. Since $\bar{A}$ is commutative, $\bar{A}$ is a field. We may regard F as a subfield of $\bar{A}$.

Consider the field automorphism $\varphi\colon a \to a^q$ of $\bar{A}$. By hypothesis, every element of $\bar{X}$ is invariant under φ. Since $\bar{A}$ is spanned as a vector space by $\bar{X}$, and since every element of the prime field F is invariant under φ, φ is the identity automorphism on $\bar{A}$. This, however, implies that $\bar{A}$ has just q elements, therefore that $\bar{A} = F$.

M is an irreducible $\bar{A}$-module, that is an irreducible F-space. Hence the F-dimension of M is 1.

Remark. Lemmas 2 and 3 are certainly well known. For instance, Lemma 2 follows immediately from the characterization by Gaschütz ('Zur Theorie der endlichen auflösbaren Gruppen', *Math. Zeitschrift* 80 (1963) 300–5, Beispiel 8) of the class of finite supersoluble groups as a locally defined formation; however, no explicit proof is to be found there. Since I am unable to cite explicit references, proofs are included here. I am indebted for these proofs to the Referee, to whom I wish to express my thanks.

Proof of Theorem 7. Let p be an odd prime, and d a divisor of $p-1$ with $d > 1$. The holomorph of a cyclic group P of order p is an extension of P by a cyclic group of order $p-1$, and therefore has a unique subgroup H of order pd, an extension of P by a cyclic group of order d. Furthermore, P is the unique minimal normal subgroup of H, and the only abnormal maximal subgroups of H are cyclic of order d. Let q be a prime divisor of d, so that in particular $q \neq p$. It is now possible to construct a group G which is a split extension of an elementary abelian q-group Q by H, such that Q is the unique minimal normal subgroup of G; for the justification of this construction, see ((**10**) Example 2). We shall show that p, d, and q may be so chosen that G fulfils the conditions required to establish Theorem 7.

Let M be any abnormal maximal subgroup of G. If $M \not\geqslant Q$ then $MQ = G$ and so $M \cap Q = 1$, by the minimality of Q. Then $M \cong G/Q \cong H$, a metacyclic and therefore supersoluble group. On the other hand, if $M \geqslant Q$ then M/Q is an abnormal maximal subgroup of G/Q, and so M/Q is cyclic of order d. Thus M is an extension of a q-group by a cyclic group of order d. It will be shown that d can be chosen to have the form $d = qr$, where r is a divisor of $q-1$. Then it will follow from Lemma 2 that M is supersoluble, so that every proper abnormal subgroup of G will be supersoluble. Since it is clear from the construction, because q divides d, that G has q-length 2, and that G admits no non-trivial decomposition as a direct product, it remains to show only that the arithmetical conditions can be satisfied.

If $n = 2$, we begin by choosing an arbitrary prime p_1. If $n > 2$, we choose first any $n-2$ distinct primes $p_2, p_3, \ldots, p_{n-1}$. By a celebrated theorem of Dirichlet, there exist primes congruent to 1 modulo $p_2 p_3 \cdots p_{n-1}$, and we choose p_1 to be one of these. We then employ Dirichlet's theorem again to ensure the existence of primes congruent to 1 modulo $p_1 p_2 p_3 \cdots p_{n-1}$, and choose p to be one of these. Finally, we may choose $d = p_1 p_2 p_3 \cdots p_{n-1}$ and $q = p_1$. Then $d = qr$, where $r = p_2 p_3 \cdots p_{n-1}$ if $n > 2$ or $r = 1$ if $n = 2$; and by the choice of p_1, r is a divisor of $q-1$. Since $|G|$ has now precisely the n distinct prime factors $p, p_1, p_2, \ldots, p_{n-1}$, the proof of Theorem 7 is complete.

It is perhaps worth mentioning that a group G in which every proper subgroup has a Sylow tower of complexion $p_1, \ldots, p_n$ is soluble (without restriction on the Sylow 2-subgroups); and that if G does not itself possess a Sylow tower of the same complexion then it belongs to a special and fully characterized class of groups. We follow Z. Janko and M. F. Newman (**8**) in calling a non-nilpotent finite group in which every proper subgroup is nilpotent an *SRI-group*. The structure of such groups has been discussed by O. J. Schmidt (**11**), by K. Iwasawa (**7**), and at length by L. Rédei (**9**). If G is an SRI-group then $|G|$ has exactly 2 distinct prime factors p, q; G has a normal Sylow subgroup corresponding to one of these primes, say p; and then the Sylow q-subgroups of G are cyclic.

N. Itô ((**6**) Proposition 2) proved that if all the proper subgroups of a finite group G are p-nilpotent, for any particular prime p, then either G is itself p-nilpotent or G is an SRI-group. From this we deduce very easily

THEOREM 8. *Suppose that every proper subgroup of the finite group G has a Sylow tower of complexion $p_1, \ldots, p_n$. Then G is soluble; and either G has itself a Sylow tower of complexion $p_1, \ldots, p_n$, or G is an SRI-group.*

Proof. G may be supposed non-nilpotent. Then, since every Sylow subgroup of G has a Sylow tower of the given complexion, it is certain that every prime factor of $|G|$ appears among $p_1, \ldots, p_n$. Also $n \geqslant 2$. It may be supposed further that every p_i is a divisor of $|G|$ $(i = 1, \ldots, n)$. If $n = 2$, every proper subgroup of G is p_2-nilpotent and Itô's theorem yields the result at once. If $n > 2$, every proper subgroup of G is p_n-nilpotent. Since $|G|$ has more than 2 distinct prime factors, G is not an SRI-group, and so Itô's theorem shows that G is p_n-nilpotent, say with p_n-complement H. Then $H < G$, and p_n does not divide $|H|$, so the hypothesis of the theorem implies that H has a Sylow tower of complexion $p_1, \ldots, p_{n-1}$. Hence G has a Sylow tower of complexion $p_1, \ldots, p_n$.

Remark. In particular, a finite group G in which every proper subgroup is supersoluble either has an ordered Sylow tower or is an SRI-group (a fact pointed out by Janko and Newman in (**8**)). Huppert gave an example ((**5**) Beispiel 4) of such a G which is non-supersoluble and in which $|G|$ has 3 distinct prime factors (the maximal possible number); G must then have an ordered Sylow tower. The structure of non-supersoluble finite groups in which every proper subgroup is supersoluble, has been studied in detail by K. Doerk; his results will be published in a paper to appear in *Math. Zeitschrift* (1966).

In view of the remark on p. 356 of (**10**), Theorem 8 has the following

COROLLARY. *If every proper self-normalizing subgroup of the finite group G has a Sylow tower of complexion $p_1, \ldots, p_n$, then G is soluble.*

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The University
Newcastle upon Tyne

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Remarks on system normalizers and Carter subgroups

JOHN S. ROSE*

In investigations into the abnormal structure of a finite soluble group, a natural problem is posed by the relationship between system normalizers and Carter subgroups. At present this eludes a satisfactory general solution. For an account of results already obtained, reference may be made to papers [1] of J. L. Alperin and [2] of R. W. Carter.

Some extensions of results of Carter on A-groups, and methods of proof different from his, are reported here. The approach adopted depends on a property of A-groups which follows from

Theorem 1. *Suppose that G is a finite soluble group, with abelian Sylow p-subgroups for some prime factor p of $|G|$. Let D be a system normalizer of G, and D_p the Sylow p-subgroup of D. Then D_p is a Sylow p-subgroup of a normal subgroup of G.*

Corollary 1. *Suppose that G is an A-group and D a system normalizer of G. Then each Sylow subgroup of D is also a Sylow subgroup of some normal subgroup of G.*

This property is perhaps rather unexpected, since D is contained in no proper normal subgroup of G. The relevance of Corollary 1 to the problem of relating system normalizers and Carter subgroups is seen by elementary considerations of *pronormality*. The definition and basic properties of pronormal subgroups are due to Professor P. Hall.

Definition. A subgroup H is pronormal in a group G if any two conjugates of H in G are already conjugate in their join.

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This isolates a useful property of Sylow subgroups in a finite group. In fact, one has

Lemma 1. *Sylow subgroups of normal subgroups are pronormal.*

Thus Corollary 1 implies that if G is an A-group, then each Sylow subgroup of a system normalizer of G is pronormal in G. From this fact, the following deduction can be made.

Corollary 2. *If G is an A-group, then the system normalizers of G are pronormal in G.*

It seems that this property of an A-group underlies Carter's results on system normalizers and Carter subgroups, for the following precise analogues of Theorems 6, 9, and 10 of [2] can be proved. (For any soluble group X, $l(X)$ denotes the nilpotent length of X.)

Theorem 2. *Suppose that G is a finite soluble group such that G and all its subgroups have pronormal system normalizers. Define $D_0 = 1$, $B_0 = G$; and, inductively, for each positive integer i, $D_i =$ a system normalizer of B_{i-1}, $B_i = N_G(D_i)$, the normalizer in G of D_i. Then*

(i) *$D_{i+1} \geq D_i$, $B_{i+1} \leq B_i$, for all i.*
(ii) *There is a Carter subgroup C of G such that $D_i \leq C \leq B_i$ for all i.*
(iii) *If $D_{i+1} = D_i$, then $D_i = C$. If $B_{i+1} = B_i$, then $B_i = C$.*
(iv) *For any Carter subgroup C of G, there is a uniquely determined sequence $D_0, B_0, D_1, B_1, \ldots$ with C as its limit.*
(v) *If $l(B_i) \geq 3$ for any particular i, then $l(B_{i+1}) \leq l(B_i) - 2$.*
(vi) *If $l(G) \leq 2n + 1$, then $B_n = C$. If $l(G) \leq 2n$, then $D_n = C$.*

In establishing his results, Carter made extensive use of four invariants, which he associated with each subgroup of a soluble group. These are not used in the proof of Theorem 2, which relies on simple properties of pronormal subgroups, rather than on more special features of A-groups. In particular, the following facts are needed.

Lemma 2. *If H is a pronormal subgroup of a group G, then the normalizer $N_G(H)$ of H in G is abnormal in G. Moreover, every subgroup of G in which H is subnormal is contained in $N_G(H)$, so that $N_G(H)$ is the subnormalizer of H in G.*

Lemma 3. *No two distinct conjugates of a pronormal subgroup are permutable.*

It follows at once from Corollary 2 and Lemma 2 that if D is a system normalizer of an A-group G, then $N_G(D)$ is the subnormalizer of D in G. This was also proved in [2] (Theorem 5), by different means, and used in the proofs of the subsequent theorems.

Details of the results described here will be given in [3].

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FINITE SOLUBLE GROUPS WITH PRONORMAL SYSTEM NORMALIZERS

By JOHN S. ROSE

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FINITE SOLUBLE GROUPS WITH PRONORMAL SYSTEM NORMALIZERS

By JOHN S. ROSE

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The abnormal structure of a finite soluble group has been the subject of several recent investigations. In these a prominent role is played by two characteristic conjugacy classes of nilpotent subgroups, the *system normalizers* and the *Carter subgroups*. The system normalizers appear naturally in P. Hall's development of an arithmetic theory of finite soluble groups. Their significance for considerations of abnormal structure derives from a striking characterization established by Hall ((**8**) Theorem 4.8), which may be expressed in the following way: *any finite soluble group G has a single conjugacy class of minimal subabnormal subgroups, and these are precisely the system normalizers of G.* (For a brief discussion of nilpotent subabnormal subgroups, see ((**11**) § 4).) This result makes available for application to problems about abnormal structure the well-developed theory of system normalizers, in particular their covering and avoidance properties ((**8**) Theorems 6.1 and 7.1). The second class of nilpotent subgroups is a discovery of R. W. Carter (**4**). He showed that *any finite soluble group G possesses nilpotent self-normalizing subgroups, all such subgroups are conjugate in G, and they are also abnormal in G.* These are called the Carter subgroups of G.

It follows at once from the results stated above that any Carter subgroup contains a system normalizer, and that any system normalizer is contained in a Carter subgroup. The problem suggests itself of describing more fully the relationship between system normalizers and Carter subgroups. This seems to be rather intractable and no complete solution has yet been found, but various interesting results bearing on the problem have appeared. For instance, J. L. Alperin (**2**) has shown that if two system normalizers D_1, D_2 are contained in a single Carter subgroup C, then D_1, D_2 are conjugate in C. Carter himself has approached the problem by restricting attention to particular classes of soluble groups. By his work (**3**) on soluble groups with self-normalizing system normalizers, it is known that the classes of system normalizers and Carter subgroups coincide in a metanilpotent group (that is, a group of nilpotent length at most 2). More recently, he has provided in (**5**) detailed information about the relationship for the classes of groups of nilpotent length 3 and of A-groups. Alperin (**2**) has given different proofs of the results for groups of nilpotent length 3.

Carter's approach is adopted in the present paper, where the results for A-groups are extended to a wider class of groups and obtained by different means. In (**5**), Carter made extensive use of four invariants which he associated with each subgroup of a soluble group. Here reliance is placed rather on the properties of *pronormal* subgroups, to be described in § 1.

The definition and basic properties of pronormal subgroups are due to Professor P. Hall, and I wish to acknowledge here my indebtedness to him and to express thanks for his many helpful suggestions. Since no account of pronormality has yet been made widely available, § 1 provides a brief survey, designed to supply the facts on which the main considerations of the paper are based.

A summary without proofs of parts of this paper has appeared already in (**12**).

All groups considered are implicitly assumed to be finite. The notation and terminology used are largely standard. If X is a non-empty subset of a group G, $N_G(X)$ denotes the normalizer of X in G, and $C_G(X)$ the centralizer of X in G. Then $C_G(G) = Z(G)$, the centre of G. It is sometimes convenient, for an arbitrary subgroup H of G, to set

$$N_H(X) = H \cap N_G(X) \quad \text{and} \quad C_H(X) = H \cap C_G(X).$$

The hypernormalizer of H in G is denoted by $N_G{}^\infty(H)$: this is the subgroup of G in which the ascending chain of subgroups $H_0 \leqslant H_1 \leqslant H_2 \leqslant \ldots$ becomes stationary, where $H_0 = H$ and, for each integer $i \geqslant 1$, $H_i = N_G(H_{i-1})$. The subgroup of G generated by X is denoted by $\{X\}$; and if Y is another subset of G, the notation $\{X, Y\}$ is used in place of $\{X \cup Y\}$. If g and x are arbitrary elements of G, we set $g^x = x^{-1}gx$ and $H^x = x^{-1}Hx$. The notation $H \lhd G$ is used to mean that H is a normal subgroup of G (not necessarily proper).

Throughout this paper, the symbol p denotes a prime number, and p' the set of all prime numbers other than p. When G is a soluble group, G_p is sometimes used to denote a Sylow p-subgroup of G, and G^p a p-complement of G. In particular, if G is nilpotent then G_p and G^p are uniquely determined, and $G = G_p \times G^p$. If G is a p-soluble group, $l_p(G)$ denotes the p-length of G, and if G is a soluble group, $l(G)$ denotes the nilpotent length of G.

A group is called *monolithic* if it has a unique minimal normal subgroup.

In the language of closure operations, a class $\mathfrak{C}$ of groups is called Q-*closed* if any homomorphic image of any group belonging to $\mathfrak{C}$ also belongs to $\mathfrak{C}$, and $\mathfrak{C}$ is called R_0-*closed* if, for any group G with normal subgroups H and K such that $G/H \in \mathfrak{C}$ and $G/K \in \mathfrak{C}$, $G/H \cap K \in \mathfrak{C}$. A class of finite soluble groups which is both Q-closed and R_0-closed is called a *formation* (W. Gaschütz (**6**)). Gaschütz has made clear the significance of formations for the theory of finite soluble groups. Of particular importance for the recognition of characteristic conjugacy classes of abnormal subgroups are the *saturated* formations: Carter subgroups appear by Gaschütz's method from the saturated formation of finite nilpotent groups.

1. Pronormal subgroups

Definition. A subgroup H is called *pronormal* in a group G if any two conjugates of H in G are already conjugate in their join.

One rather obvious but helpful observation may be made. In proving a subgroup H pronormal in a group G, it is enough to show that for any element x of G there exists an element y of $\{H, H^x\}$ such that $H^x = H^y$; for then, if H^a, H^b are any two conjugates of H in G $(a, b \in G)$, $H^{ba^{-1}} = H^c$ for some c in $\{H, H^{ba^{-1}}\}$, and so $H^b = H^{ca} = (H^a)^{a^{-1}ca}$ with

$$a^{-1}ca = c^a \in \{H^a, H^b\}.$$

All normal subgroups and all abnormal subgroups of a group are clearly included among its pronormal subgroups. Further examples are provided by

1.1. *Sylow subgroups of normal subgroups are pronormal. Hall subgroups of soluble normal subgroups are pronormal.*

The basic properties of pronormal subgroups may be verified in a few lines by straightforward arguments.

1.2. *If H is pronormal in G and $H \leqslant L \leqslant G$, then H is pronormal in L.*

1.3. *If $K \lhd G$ and $K \leqslant H \leqslant G$, then H is pronormal in G if and only if H/K is pronormal in G/K.*

1.4. *If H is pronormal in G and $K \lhd G$, then HK is pronormal in G and $N_G(HK) = N_G(H)K$.*

The next two statements are of particular importance for the subsequent theory. They show that all pronormal subgroups possess certain special properties known for Sylow subgroups.

1.5. *H is both pronormal and subnormal in G if and only if $H \lhd G$.*

1.6. *If H is pronormal in G, then $N_G(H)$ is abnormal in G. Moreover, every subgroup of G in which H is subnormal is contained in $N_G(H)$, so that $N_G(H)$ is the subnormalizer of H in G.*

The following will also be needed.

1.7. *No two distinct conjugates of a pronormal subgroup are permutable.*

Proof. Suppose that H is pronormal in G, and that H^x is permutable with H, for some x in G. Then $H^x = H^y$, for some y in $\{H, H^x\} = HH^x$. Now y may be expressed in the form $y = h_1 h_2{}^x$, with h_1, h_2 in H. Therefore $H^x = H^{h_1 x^{-1} h_2 x} = H^{x^{-1} h_2 x}$, from which it follows that $H = H^{x^{-1} h_2}$ and $H^x = H$.

1.8. *Suppose that H and K are subgroups of G such that $K \leqslant N_G(H)$. If H and K are both pronormal in G, then HK is pronormal in G.*

Proof. Let $J = HK$. Since H is pronormal in G, for any x in G, $H = H^{xy}$ for some y in $\{H, H^x\} \leqslant \{J, J^x\}$. Since $H \lhd J$, it follows that $H \lhd \{J, J^{xy}\}$. By hypothesis, K is pronormal in G, so that $K = K^{xyz}$ for some z in $\{K, K^{xy}\} \leqslant \{J, J^{xy}\} \leqslant \{J, J^x\}$. It follows from $H \lhd \{J, J^{xy}\}$ that $H = H^{xyz}$, and so $J = HK = H^{xyz}K^{xyz} = J^{xyz}$, with yz in $\{J, J^x\}$. Hence J is pronormal in G.

2. A property of A-groups

In any finite soluble group, the Carter subgroups are abnormal, and so in particular pronormal. In a metanilpotent group, the classes of system normalizers and Carter subgroups coincide, so that then the system normalizers are also certainly pronormal. However, the symmetric group of degree 4 demonstrates that system normalizers in general need not be pronormal.

The results of § 1 show that if a soluble group G has pronormal system normalizers, the abnormal structure of G has various agreeable features. In particular, proposition 1.4 shows that normalizers of system normalizers of G have the property of homomorphic invariance (under homomorphisms of G), the importance of which has been stressed by Alperin (**1**). (System normalizers themselves always possess the property of homomorphic invariance (Hall ((**8**) Theorem 7.3)), but not so their normalizers. This is a major source of difficulty in attempts to investigate generally the relationship between system normalizers and Carter subgroups.) Proposition 1.6 shows further that the normalizer of a system normalizer D of G is also the subnormalizer of D, and is abnormal in G. These properties were established by Carter for A-groups in (**5**). It will now be proved that all A-groups actually have pronormal system normalizers.

The following lemma is a well-known result. It is a particular case of Theorem 1.2.6 of P. Hall and G. Higman (**9**), and has also been proved in another way by B. Huppert ((**10**) Satz 17).

LEMMA 2.1. *Suppose that the p-soluble group G has abelian Sylow p-subgroups, where p is a prime factor of $|G|$. Then $l_p(G) = 1$.*

THEOREM 2.2. *Suppose that G is a finite soluble group with abelian Sylow p-subgroups, for some prime p. Let D be a system normalizer of G. Then the Sylow p-subgroup D_p of D is also a Sylow p-subgroup of some normal subgroup of G.*

Proof. We use induction on $|G|$. Let K be the largest normal p'-subgroup of G. Then G/K has abelian Sylow p-subgroups, and DK/K is a system normalizer of G/K, with Sylow p-subgroup D_pK/K. If $K > 1$, the induction hypothesis implies that there is a normal subgroup L/K of G/K such that D_pK/K is a Sylow p-subgroup of L/K. Since

$$|L : D_p| = |L : D_pK|\,|D_pK : D_p| = |L/K : D_pK/K|\,|K|,$$

D_p is a Sylow p-subgroup of L; and $L \lhd G$.

Thus we may assume that $K = 1$. By Lemma 2.1, it follows that G has a normal Sylow p-subgroup, P say. We know then that

$$D_p = P \cap N_G(R)$$

for some p-complement R of G (Hall ((**8**) Theorem 3.3)). Therefore $D_p \lhd N_G(R)$, since $P \lhd G$. Furthermore, since P is by hypothesis abelian, $D_p \lhd P$. Therefore $D_p \lhd PN_G(R) = G$; and D_p is a Sylow p-subgroup of itself. This completes the induction argument.

This theorem shows in particular that A-groups have the following curious property. (We recall that a system normalizer of an arbitrary soluble group G is contained in no proper normal subgroup of G.)

COROLLARY 2.3. *Suppose that G is an A-group and that D is a system normalizer of G. Then each Sylow subgroup of D is also a Sylow subgroup of some normal subgroup of G.*

By 1.1, this shows that if G is an A-group, each Sylow subgroup of a system normalizer of G is pronormal in G. Then application of 1.8 yields immediately

COROLLARY 2.4. *The system normalizers of an A-group G are pronormal in G.*

Theorem 2.2 is not generally true in the absence of the condition on the Sylow p-subgroups of G, even when G is metanilpotent—although in that case the system normalizers are abnormal, and so in particular pronormal in G.

EXAMPLE 2.5. *A finite metanilpotent group G may have a system normalizer D, and Sylow p-subgroup D_p of D, such that D_p is not a Sylow subgroup of a normal subgroup of G.*

Construction. Let G be the wreath product (according to regular representations) of a dihedral group of order 8 by a cyclic group of odd prime order q. Then G has nilpotent length 2, and may be expressed as

$$G = HQ,$$

where $H = \mathrm{Dr}\prod_{i=1}^{q}\{x_i, y_i\}$, $Q = \{u\}$, and $x_i^4 = y_i^2 = 1 = (x_iy_i)^2$, $u^q = 1$, $x_i^u = x_{i+1}, y_i^u = y_{i+1}$ for all i (the suffixes interpreted modulo q). H and Q define a Sylow system of G, and if D is the corresponding system normalizer, its Sylow 2-subgroup is

$$D_2 = H \cap N_G(Q) = C_H(Q),$$

since $H \lhd G$ and therefore $D_2 \lhd N_G(Q)$, $Q \lhd N_G(Q)$, and $D_2 \cap Q = 1$. It is easy to show from this that

$$D_2 = \{x_1x_2 \ldots x_q, y_1y_2 \cdots y_q\},$$

a dihedral group of order 8. Then

$$(y_1y_2 \cdots y_q)^{x_1} = x_1^2(y_1y_2 \cdots y_q) \notin D_2,$$

since $x_1^2 \notin D_2$. Therefore $D_2 \ntriangleleft G$. However, as a subgroup of the 2-group $H \lhd G$, D_2 is subnormal in G.

If D_2 were a Sylow subgroup of a normal subgroup of G then, by 1.1, D_2 would be pronormal in G; and so, by 1.5, D_2 would be normal in G. Since this is false, D_2 cannot be a Sylow subgroup of a normal subgroup of G.

3. The class $\mathfrak{X}$ of finite soluble groups with pronormal system normalizers

It will be convenient to denote by a special symbol the class of finite soluble groups with pronormal system normalizers, and the symbol $\mathfrak{X}$ is used for this purpose throughout the present paper. It was pointed out at the beginning of § 2 that $\mathfrak{X}$ contains the class of finite metanilpotent groups, and it has been proved that $\mathfrak{X}$ contains also the class of A-groups (Corollary 2.4). Closure properties of $\mathfrak{X}$ will now be investigated, and it will be shown that $\mathfrak{X}$ is a formation.

We begin with a general property of system normalizers.

LEMMA 3.1. *If G is a finite soluble group, D is a system normalizer of G, and $H \lhd G$, $K \lhd G$, then $(DH) \cap (DK) = D(H \cap K)$.*

Proof. We may assume, without loss of generality, that $H > 1$, $K > 1$, and $H \cap K = 1$. We use induction on $|G|$. Let H_1 be a minimal normal subgroup of G contained in H. Then, since DH_1/H_1 is a system normalizer of G/H_1, the induction hypothesis implies that

$$(DH/H_1) \cap (DH_1K/H_1) = DH_1/H_1,$$

so that

$$(DH) \cap (DK) = (DH_1) \cap (DK).$$

Therefore we may suppose that $H = H_1$, that is that H is a minimal normal subgroup of G. In a similar way, we may suppose that K is a minimal normal subgroup of G.

By the covering property of D, we may suppose further that both $H/1$ and $K/1$ are eccentric chief factors of G. Then HK/H is an eccentric chief factor of G, and so by the avoidance property of D,

$$D \cap (HK) = 1.$$

Then if $d_1h = d_2k$, with $d_1, d_2 \in D$, $h \in H$, $k \in K$, it follows that

$$d_2^{-1}d_1 = kh^{-1} \in D \cap (HK) = 1,$$

and so $h = k \in H \cap K = 1$. Therefore $(DH) \cap (DK) = D$. This establishes the lemma.

It is perhaps worth while to point out that the analogous result for Carter subgroups is also true, although this will not be used in the sequel.

Corollary 3.2. *If G is a finite soluble group, C is a Carter subgroup of G, and $H \lhd G$, $K \lhd G$, then $(CH) \cap (CK) = C(H \cap K)$.*

Proof. Again we may assume that $H > 1$, $K > 1$, and $H \cap K = 1$. Again we use induction on $|G|$. Let H_1 be a minimal normal subgroup of G contained in H. Since CH_1/H_1 is a Carter subgroup of G/H_1, the induction hypothesis implies that

$$(CH/H_1) \cap (CH_1K/H_1) = CH_1/H_1,$$

so that

$$(CH) \cap (CH_1K) = CH_1. \tag{1}$$

Similarly, if K_1 is a minimal normal subgroup of G contained in K,

$$(CK) \cap (CHK_1) = CK_1. \tag{2}$$

From (1) and (2),

$$(CH) \cap (CK) = (CH_1) \cap (CK_1). \tag{3}$$

Now H_1K_1 $(= H_1 \times K_1)$ is abelian, and C is nilpotent, so that CH_1K_1 is metanilpotent. Moreover, C is a Carter subgroup of CH_1K_1. Therefore, by Carter's result on metanilpotent groups, C is also a system normalizer of CH_1K_1. Lemma 3.1 may now be applied to the group CH_1K_1 to yield

$$(CH_1) \cap (CK_1) = C.$$

Then from (3),

$$(CH) \cap (CK) = C.$$

This completes the induction argument.

The following simple lemma will be used in showing that $\mathfrak{X}$ is a formation. A proof may be found in (**11**).

LEMMA 3.3. *Suppose that $G = HK$, where H, K are nilpotent subgroups of the finite group G, and $K \lhd G$. Then the hypernormalizer $N_G^{\infty}(H)$ of H in G is a Carter subgroup of G.*

THEOREM 3.4. *$\mathfrak{X}$ is a formation.*

Proof. The fact that a homomorphic image of a group in $\mathfrak{X}$ also belongs to $\mathfrak{X}$ follows at once from the homomorphic invariance of system normalizers and from 1.3 and 1.4. Thus $\mathfrak{X}$ is Q-closed.

To establish that $\mathfrak{X}$ is a formation, it is necessary to prove also that if $H \lhd G$, $K \lhd G$, and $G/H \in \mathfrak{X}$, $G/K \in \mathfrak{X}$, then $G/H \cap K \in \mathfrak{X}$. We may assume that $H \cap K = 1$, and that $H > 1$, $K > 1$. We use induction on $|G|$. Let D be a system normalizer of G. By hypothesis, DH/H is pronormal in G/H and DK/K is pronormal in G/K. We wish to prove that D is pronormal in G.

Let H_1 be a minimal normal subgroup of G contained in H. Since $(G/H_1)/(H/H_1) \in \mathfrak{X}$ and $(G/H_1)/(H_1K/H_1) \in \mathfrak{X}$, the induction hypothesis implies that $G/H_1 \in \mathfrak{X}$. Thus we may assume that $H = H_1$, and therefore in particular that H is abelian.

We consider the canonical isomorphism

$$DHK/K \cong DH/(DH) \cap K.$$

By Lemma 3.1, $(DH) \cap K = D \cap K$. It follows from this that the image of DK/K in the isomorphism is $D/(D \cap K)$. Since DK/K is pronormal in DHK/K, the isomorphism shows that $D/(D \cap K)$ is pronormal in $DH/(D \cap K)$. Therefore D is pronormal in DH.

Let $E = N_{DH}(D)$. Since D is pronormal in DH, it follows by 1.6 that $E = N_{DH}^{\infty}(D)$. Therefore, since H is abelian and by Lemma 3.3, E is a Carter subgroup of DH. Now DH is pronormal in G, so that, for any x in G,

$$(DH)^x = (DH)^y, \text{ for some } y \text{ in } \{DH, (DH)^x\} = H\{D, D^x\}.$$

Without loss of generality, we may suppose that $y \in \{D, D^x\} = J$, say. E^x and E^y are Carter subgroups of $(DH)^x = D^xH$, and so there exists an element u of D^xH such that $E^x = E^{yu}$. Then

$$D^x \lhd E^x \quad \text{and} \quad D^{yu} \lhd E^{yu} = E^x,$$

so that D^x and D^{yu} are permutable subgroups of G. Therefore

$$D^xK = (DK)^x \quad \text{and} \quad D^{yu}K = (DK)^{yu}$$

are also permutable. On the other hand, these are conjugate pronormal subgroups of G, and so, by 1.7,

$$D^xK = D^{yu}K.$$

Also

$$D^xH = D^yH = D^{yu}H,$$

since $u \in D^xH$. Therefore, by Lemma 3.1,

$$D^x = D^{yu}.$$

Now $yu \in JH = HJ$, and so $yu = hj$, for some h in H and j in J. Then, since D is pronormal in DH,

$$D^x = D^{hj} = D^{vj}, \text{ for some } v \text{ in } \{D, D^h\};$$

and

$$D^h = D^{xj^{-1}} \leqslant \{D^x, j\} \leqslant J.$$

Hence $v \in J$; and $D^x = D^{vj}$ with $vj \in J$. Therefore D is pronormal in G, and $G \in \mathfrak{X}$. The result follows by induction.

We observe next that the property of belonging to $\mathfrak{X}$ is not in general inherited by subgroups. The example described by Carter (**3**) to show that a subgroup of a soluble group with self-normalizing system normalizers need not have self-normalizing system normalizers serves also for the present purpose.

EXAMPLE 3.5. *A group G in $\mathfrak{X}$ may have a subgroup N not in $\mathfrak{X}$.*

Construction. Let k denote the Galois field GF(3^3). Then G is the group of all mappings of k into itself of the form

$$x \to ax^{3^r} + b, \tag{1}$$

where $a, b \in k$, $a \neq 0$, and $r = 0, 1, 2$. G is soluble, of order $2.3^4.13$, and it is shown in (**3**) that the system normalizers of G are self-normalizing, and hence abnormal, in G. Therefore $G \in \mathfrak{X}$. The subset of G consisting of mappings in which a is a square in k forms a normal subgroup N of G of index 2; and it is shown that the system normalizers of N have order 3.

The mapping (1) will be denoted by $\varphi(a, b; r)$. Then $\bar{D} = \{\varphi(1, 0; 1)\}$ is a system normalizer of N: for $|\bar{D}| = 3$, and $\bar{D}$ normalizes a Sylow system of N. ($\bar{D}$ normalizes any Sylow 3-subgroup of N which contains it, and also $\{\varphi(a, 0; 0): a \in k, a \neq 0\} \cap N$, which is a Sylow 13-subgroup of N.)

Now

$$K = \{\varphi(1, b; 0): b \in k\} \lhd G,$$

$|K| = 3^3$, and $\bar{D} \nleqslant K$. Therefore $\bar{D}K$ is a Sylow 3-subgroup of N. If $\bar{D}$

were pronormal in N then, by 1.5, $\bar{D}$ would be normal in $\bar{D}K$. This is however false, since

$$\varphi(1,b;\,0)^{-1}\varphi(1,0;\,1)\varphi(1,b;\,0) = \varphi(1,b-b^3;\,1);$$

this belongs to $\bar{D}$ only if $b^3 = b$, whereas b may be chosen as an element outside the prime subfield of k. Thus $G \in \mathfrak{X}$ but $N \notin \mathfrak{X}$.

4. System normalizers and Carter subgroups

In (**5**), Carter gave a systematic construction for the Carter subgroups of an A-group, by means of system normalizers and their normalizers. In the present section, it will be shown that the property of A-groups established in Corollary 2.4 is relevant to this construction, which will be proved to hold for a wider class of groups by using the properties of pronormal subgroups from § 1.

First, however, we shall extend another result of Carter. He proved in (**5**) that if D is a system normalizer of an A-group G, and H is any subgroup of G containing D, then D is contained in a system normalizer of H. He remarked that it was not known whether or not this statement held for all soluble groups; although he had proved in (**3**) (Theorem 3.1) that $|D|$ certainly divides the order of the system normalizers of H, for arbitrary soluble G. More recently, Alperin (**1**) has constructed a soluble group of nilpotent length 3, which has a system normalizer D and a maximal subgroup M containing D such that D normalizes no Sylow system of M. Thus Carter's result for A-groups is false for general soluble groups, but the following extension holds.

THEOREM 4.1. *If $G \in \mathfrak{X}$, D is a system normalizer of G, and $G \geqslant H \geqslant D$, then D is contained in a system normalizer of H.*

Proof. Let $H = H_0 < H_1 < \dots < H_r = G$ be a maximal chain of subgroups connecting H to G, and proceed by induction on r. The result is trivial if $r = 0$, and so it may be assumed that $r > 0$. The induction hypothesis implies that there is a system normalizer E_1 of H_1 with $D \leqslant E_1$. Now H is maximal in H_1, and there are two possibilities.

(i) $H \lhd H_1$. In this case, if $\mathfrak{S}$ is a Sylow system of H_1 normalized by E_1, and hence also by D, then $\mathfrak{S}$ is reducible into H. Thus $\mathfrak{S} \cap H$ is a Sylow system of H normalized by D. Since D lies in H, it follows that D lies in a system normalizer of H.

(ii) H is abnormal in H_1. Then, if E is a system normalizer of H, E is subabnormal in H_1 and so E contains some system normalizer of H_1, say $E \geqslant E_1{}^x$ with x in H_1. Then $E \geqslant D^x$, and since $G \in \mathfrak{X}$, $D^x = D^y$ for some y in $\{D, D^x\} \leqslant H$. Thus $D \leqslant E^{y^{-1}}$, a system normalizer of H.

The result follows by induction on r.

The following lemma is a consequence of Alperin's result on the conjugacy of system normalizers contained in a single Carter subgroup, but a short independent proof is included here.

LEMMA 4.2. *Suppose that* $G \in \mathfrak{X}$. *Then each Carter subgroup of* G *contains just one system normalizer of* G.

Proof. Let C be a Carter subgroup of G. Then C certainly contains at least one system normalizer of G, by the characterization of system normalizers as the minimal subabnormal subgroups. Suppose that C contains system normalizers D, D^x of G $(x \in G)$. Let $B = N_G(D)$. Then $B^x = N_G(D^x)$. Since C is nilpotent, D is subnormal in C, so that, by 1.6, $C \leqslant B$; and likewise $C \leqslant B^x$. But since C is abnormal in G, this implies that $B^x = B$. Hence $x \in N_G(B) = B$, since $C \leqslant B$. Therefore $D^x = D$.

We come now to the extensions of Carter's results ((**5**) Theorems 6, 9, 10). The proofs follow closely those of Carter, but are based on the properties of pronormal subgroups rather than on more special features of A-groups.

THEOREM 4.3. *Suppose that* G *is a finite soluble group such that* G *and all its subgroups belong to* $\mathfrak{X}$. *(In particular, this condition is satisfied if* G *is an* A*-group.) Define* $D_0 = 1$, $B_0 = G$; *and inductively, for each positive integer* i, $D_i =$ *a system normalizer of* B_{i-1}, $B_i = N_G(D_i)$. *Then*

(i) $D_{i+1} \geqslant D_i$, $B_{i+1} \leqslant B_i$, *for all* i;

(ii) *there is a Carter subgroup* C *of* G *such that* $D_i \leqslant C \leqslant B_i$, *for all* i;

(iii) *if* $D_{i+1} = D_i$ *then* $D_i = C$; *if* $B_{i+1} = B_i$ *then* $B_i = C$;

(iv) *for any Carter subgroup* C *of* G, *there is a uniquely determined sequence* $D_0, B_0, D_1, B_1, \ldots$, *with* C *as its limit*;

(v) *if* $l(B_i) \geqslant 3$ *for any particular* i, *then* $l(B_{i+1}) \leqslant l(B_i) - 2$;

(vi) *if* $l(G) \leqslant 2n+1$ *then* $B_n = C$; *if* $l(G) \leqslant 2n$ *then* $D_n = C$.

Proof. (i) We use induction on i. The assertion is trivial for $i = 0$, so we suppose that $i > 0$, and assume the result true for all $j < i$:

$$D_{j+1} \geqslant D_j, \quad B_{j+1} \leqslant B_j.$$

By hypothesis, D_i is pronormal in B_{i-1}, so that, by 1.6, $N_{B_{i-1}}(D_i)$ is abnormal in B_{i-1}. Then, since $N_{B_{i-1}}(D_i) \leqslant N_G(D_i) = B_i \leqslant B_{i-1}$, by the induction hypothesis, it follows that B_i is abnormal in B_{i-1}. Therefore D_{i+1} is subabnormal in B_{i-1}, and hence D_{i+1} contains some system normalizer of B_{i-1}, say $D_{i+1} \geqslant D_i^b$, where $b \in B_{i-1}$. This implies that

$$D_i^b \leqslant B_i = N_G(D_i),$$

so that D_i^b normalizes D_i. On the other hand, D_i is pronormal in B_{i-1}, and so, by 1.7, $D_i^b = D_i$. Hence $D_{i+1} \geqslant D_i$.

In order to complete the proof of (i), we use a separate induction argument to show that, for fixed i, $B_{i+1} \leqslant B_j$ for all $j \leqslant i$. This is trivial for $j = 0$, so we suppose that $j > 0$ and assume that $B_{i+1} \leqslant B_{j-1}$. Now $D_j \leqslant \ldots \leqslant D_i \leqslant D_{i+1}$, and since D_{i+1} is nilpotent, D_j is subnormal in B_{i+1}. Since $B_{i+1} \leqslant B_{j-1}$ and D_j is pronormal in B_{j-1}, it follows by 1.6 that $B_{i+1} \leqslant N_{B_{j-1}}(D_j) \leqslant N_G(D_j) = B_j$. This completes the induction argument on j. It follows that $B_{i+1} \leqslant B_i$, and this completes the induction argument on i.

(ii) Again we use induction on i. The assertion is certainly true when $i = 0$, so we assume that $i > 0$ and that there exists a Carter subgroup C of G such that $D_{i-1} \leqslant C \leqslant B_{i-1}$. Since D_i is a system normalizer of B_{i-1}, D_i is contained in some Carter subgroup C^* of B_{i-1}. Now since $C \leqslant B_{i-1}$, C is also a Carter subgroup of B_{i-1}, and therefore every Carter subgroup of B_{i-1} is a Carter subgroup of G. Hence C^* is a Carter subgroup of G. By hypothesis, D_i is pronormal in B_{i-1}, and so, by 1.6, $D_i \lhd C^*$. Therefore $D_i \leqslant C^* \leqslant N_G(D_i) = B_i$. This completes the induction argument.

(iii) If $D_{i+1} = D_i$ then D_i is a system normalizer of B_i and $D_i \lhd B_i$. This is possible only if $D_i = B_i$, and then, by (ii), $D_i = C$.

If $B_{i+1} = B_i$ then D_{i+1} is a system normalizer of B_i and $D_{i+1} \lhd B_i$. Hence $D_{i+1} = B_i$, and then, by (ii), $B_{i+1} = B_i = C$.

(iv) This follows from Lemma 4.2, since C is a Carter subgroup of each term B_i of a sequence $D_0, B_0, D_1, B_1, \ldots$ having C as its limit. Thus, given any B_{i-1}, D_i is uniquely determined; then B_i is uniquely determined as the normalizer in G of D_i; and so on.

(v) B_{i+1} is the normalizer of a system normalizer of B_i, and $B_{i+1} \leqslant B_i$, by (i). Therefore B_{i+1} is one of the subgroups $B_1(B_i)$. Hence it is enough to consider here the case $i = 0$.

We prove by induction on $l(G)$ that if $l(G) \geqslant 3$ then $l(B_1) \leqslant l(G)-2$. If $l(G) = 3$ then Carter's results on groups of nilpotent length 3 ((**5**) Theorems 2, 3; or Alperin (**2**)) show that D_1 has a subnormalizer in G, which is in fact the unique Carter subgroup C of G containing D_1. By 1.6, $B_1 = C$, so that $l(B_1) = 1$.

Now suppose that $l(G) > 3$, and let F be the Fitting subgroup of G. Then $l(G/F) = l(G)-1$. D_1F/F is a system normalizer of G/F, and, by 1.4, B_1F/F is one of the subgroups $B_1(G/F)$. By the induction hypothesis, since $l(G/F) \geqslant 3$, $l(B_1F/F) \leqslant l(G/F)-2$. Since $B_1F/F \cong B_1/(B_1 \cap F)$ and $B_1 \cap F$ is nilpotent, $l(B_1) \leqslant l(B_1F/F)+1 \leqslant l(G)-2$, as required.

(vi) The assertions follow readily from (v), by induction on n, exactly as in the proof of Theorem 10 of (**5**).

Inspection of the proofs shows that Theorem 4.3 is valid under the assumption that G and all its subabnormal subgroups belong to $\mathfrak{X}$.

Whether the theorem remains true merely under the hypothesis that $G \in \mathfrak{X}$, or even more generally, is not known.

It is relevant to ask whether Theorem 4.3 is applicable to groups other than A-groups and (trivially) metanilpotent groups. We may observe here that if $G = G_1 \times G_2$, where G_1 is an A-group and G_2 is metanilpotent, and if H is a subgroup of G, then $H/(H \cap G_1) \cong HG_1/G_1$, which is metanilpotent, and $H/(H \cap G_2) \cong HG_2/G_2$, which is an A-group; therefore, by Theorem 3.4, $H \in \mathfrak{X}$. Hence Theorem 4.3 is applicable to direct products of A-groups and metanilpotent groups. Some other groups for which Theorem 4.3 is valid will be found in Corollary 5.4.

5. Soluble groups with special Sylow structure

In this section a number of different criteria will be established for a group, whose Sylow structure is in some way special, to belong to $\mathfrak{X}$. We begin by proving that a certain condition is necessary and sufficient for a group G to belong to $\mathfrak{X}$, when G has a normal Sylow subgroup P such that $G/P \in \mathfrak{X}$. For this purpose the following lemma is needed.

Lemma 5.1. *Suppose that G is a finite soluble group with a normal Sylow p-subgroup P. Let D be a system normalizer of G, and let G^p be any p-complement of G containing the unique p-complement D^p of D. Then D^p is a system normalizer of G^p.*

Proof. $G^p \cong G/P$, and this isomorphism induces a one-to-one correspondence between the subgroups of G^p and the subgroups of G/P, in which system normalizers correspond to system normalizers. DP/P is a system normalizer of G/P. Moreover, $D^pP = DP$. Since D^pP/P corresponds to D^p in the natural isomorphism $G^p \cong G/P$, the result is proved.

Theorem 5.2. *Suppose that the finite soluble group G has a normal Sylow p-subgroup P such that $G/P \in \mathfrak{X}$. Let D be a system normalizer of G. Then $G \in \mathfrak{X}$ if and only if $D_p \lhd C_P(D^p)$.*

Proof. Suppose first that $G \in \mathfrak{X}$. Then for any x in G, there exists an element y of $J = \{D, D^x\}$ for which $D^x = D^y$. In particular this implies that ${D_p}^x = {D_p}^y$. If x is chosen to be an element of $C_P(D^p)$, then clearly $J = \{D_p, {D_p}^x\} \times D^p$, and in this case there exists an element z of $\{D_p, {D_p}^x\}$ such that ${D_p}^x = {D_p}^z$. Certainly $D_p \leqslant C_P(D^p)$, and the preceding remarks show that D_p is pronormal in $C_P(D^p)$. Since $C_P(D^p)$ is a p-group, D_p is also subnormal in $C_P(D^p)$, and then 1.5 shows that $D_p \lhd C_P(D^p)$. This proves the necessity of the condition for G to belong to $\mathfrak{X}$.

Now suppose conversely that $D_p \lhd C_P(D^p)$. Given an element x of G, let $J = \{D, D^x\}$. Put $E = D^p$. Then E and E^x are p'-subgroups of J, and therefore, by Hall's theorem, there exists an element y of J such that E and E^{xy} lie in the same p-complement of J. If Q is any p-complement of G containing $\{E, E^{xy}\}$, then, by Lemma 5.1, E and E^{xy} are system normalizers of Q; so that there exists an element u of Q with $E^{xy} = E^u$. Since $Q \simeq G/P \in \mathfrak{X}$, E is pronormal in Q. Hence there exists an element z of $\{E, E^u\}$ such that $E^{uz} = E$. Furthermore, $E^u \leqslant \{E^x, y\} \leqslant J$. Therefore $E^{xyz} = E$ and $yz \in J$. It follows from this that we may assume that $E^x = E$: for if from $E^{xyz} = E$ we can show that $D^{xyzt} = D$, for some t in $\{D, D^{xyz}\}$, then $t \in \{D, D^x, yz\} = J$, so that $D^{xyzt} = D$ with yzt in J.

Thus we suppose now that $x \in N_G(E)$. If Q, Q_1 are p-complements of G which contain E, then, by Lemma 5.1, E is a system normalizer of both Q and Q_1; and by the isomorphisms

$$Q \cong G/P \cong Q_1, \tag{1}$$

it follows that

$$N_Q(E) \cong N_{Q_1}(E). \tag{2}$$

Now $N_G(E)$ has normal Sylow p-subgroup $N_P(E)$, and

$$N_G(E) = N_P(E)V, \tag{3}$$

where V is any p-complement of $N_G(E)$. Since E is a normal p'-subgroup of $N_G(E)$, it follows by Hall's theorem that $E \leqslant V$. By Hall's theorem again, $V \leqslant Q_1$ for some p-complement Q_1 of G, and then

$$V = Q_1 \cap N_G(E) = N_{Q_1}(E). \tag{4}$$

Let Q be the p-complement of G belonging to a Sylow system of G normalized by D. Then Q is the unique p-complement of $N_G(Q)$, and therefore, since $D \leqslant N_G(Q)$, $E \leqslant Q$. By the isomorphism (2), together with (3) and (4), this implies that

$$N_G(E) = N_P(E)N_Q(E). \tag{5}$$

Next, $D_p = P \cap N_G(Q)$ (Hall ((8) Theorem 3.3)), and so $D_p \lhd N_G(Q)$. In particular, D_p is normalized by Q, and therefore

$$N_Q(E) \leqslant N_G(D_p). \tag{6}$$

Moreover, since $P \lhd G$ and $P \cap E = 1$, $N_P(E) = C_P(E)$. By hypothesis, $C_P(E) \leqslant N_G(D_p)$. Hence, by (5) and (6),

$$N_G(E) \leqslant N_G(D_p).$$

Since $x \in N_G(E)$, it follows that also $x \in N_G(D_p)$. Therefore

$$D^x = D_p{}^x \times E^x = D_p \times E = D.$$

Hence D is pronormal in G. This establishes the sufficiency.

As an immediate deduction, we have

COROLLARY 5.3. *If the finite soluble group G has an abelian normal Sylow p-subgroup P such that $G/P \in \mathfrak{X}$, then $G \in \mathfrak{X}$.*

By repeated application of Corollary 5.3, and by making use of the fact that metanilpotent groups belong to $\mathfrak{X}$, we derive

COROLLARY 5.4. *If the Sylow tower group G has a Sylow series of complexion $p_1, p_2, \ldots, p_r$, where $r \geqslant 3$, and if the Sylow p_i-subgroups of G are abelian for $1 \leqslant i \leqslant r-2$, then $G \in \mathfrak{X}$.*

Corollary 5.4 yields a class of groups, all of whose subgroups belong to $\mathfrak{X}$, and to which therefore Theorem 4.3 applies. These groups are in general not direct products of A-groups and metanilpotent groups. We shall see in Example 5.6 that if, in the statement of Corollary 5.4, $r-2$ is replaced by $r-3$, the result fails to hold.

As a further application of Theorem 5.2, we shall establish a necessary and sufficient condition for a group G of p-length 1 for every prime factor p of $|G|$ to belong to $\mathfrak{X}$.

THEOREM 5.5. *Suppose that the finite soluble group G has p-length 1 for every prime factor p of $|G|$. Then $G \in \mathfrak{X}$ if and only if for any nilpotent subabnormal subgroup H of G there is a system normalizer D of G such that $D \lhd H$.*

Proof. Suppose first that $G \in \mathfrak{X}$, and let H be a nilpotent subabnormal subgroup of G. Since H is subabnormal in G, H contains some system normalizer D of G; and since H is nilpotent, D is subnormal in H. But, by hypothesis, D is also pronormal in H, so that, by 1.5, $D \lhd H$.

Now suppose conversely that G satisfies the condition given in the statement of the theorem. We prove that $G \in \mathfrak{X}$ by induction on $|G|$. If $K \lhd G$ and H/K is a nilpotent subabnormal subgroup of G/K, let E be a system normalizer of H. Then E is a nilpotent subabnormal subgroup of G, so that by hypothesis there is a system normalizer D of G such that $D \lhd E$. It follows that $DK \lhd EK$, and, since H/K is nilpotent, the covering properties of E show that $EK = H$. Thus DK/K is a system normalizer of G/K such that $DK/K \lhd H/K$. Since the class of soluble groups with p-length 1 for every prime factor p of their orders is Q-closed, we see that quotient-groups of G possess the same properties as those assumed for G. Since $\mathfrak{X}$ is a formation (Theorem 3.4), the induction hypothesis allows us to suppose that G is monolithic. The unique minimal normal subgroup of G is a p-group for some prime p, and it follows from the hypothesis $l_p(G) = 1$ that G has a normal Sylow p-subgroup, P say. By the induction hypothesis, $G/P \in \mathfrak{X}$.

Suppose that D is a system normalizer of G, and let $P_0 = C_P(D^p)$. Further, let $D^* = \{P_0, D^p\} = P_0 \times D^p$. Then D^* is nilpotent, and $D^* \leqslant D^*P = DP$. Since D is a system normalizer of G, and $P \lhd G$, DP is subabnormal in G. By Lemma 3.3, $\bar{D} = N_{D^*P}{}^{\infty}(D^*)$ is a Carter subgroup of D^*P. Therefore $\bar{D}$ is a nilpotent subabnormal subgroup of G, and so by hypothesis $D^x \lhd \bar{D}$ for some x in G. Set $y = x^{-1}$. Then

$$N_G(D) \geqslant \bar{D}^y \geqslant (D^*)^y = P_0{}^y \times (D^p)^y.$$

D^p is characteristic in D, and therefore $N_G(D^p) \geqslant N_G(D)$. From this it follows that $N_G(D^p) \geqslant P_0{}^y$, and consequently

$$P_0{}^y \leqslant P \cap N_G(D^p) = C_P(D^p) = P_0.$$

By equality of orders, $P_0{}^y = P_0$. Hence $N_G(D) \geqslant P_0$. However, D_p is characteristic in D, so that $N_G(D_p) \geqslant N_G(D)$. Therefore $P_0 \leqslant N_G(D_p)$, that is $D_p \lhd C_P(D^p)$. By Theorem 5.2, this implies that $G \in \mathfrak{X}$. This completes the induction argument.

By Lemma 2.1, all A-groups satisfy the conditions of Theorem 5.5, so that the theorem yields again the result of Corollary 2.4.

We shall now construct a Sylow tower group, of nilpotent length 3, which does not belong to $\mathfrak{X}$. This will show in particular that a group G may have a normal Sylow subgroup P such that $G/P \in \mathfrak{X}$ (indeed, such that G/P has abnormal system normalizers) but $G \notin \mathfrak{X}$. It will show too that the class of all soluble groups G in which $l_p(G) = 1$ for every prime factor p of $|G|$ is not contained in $\mathfrak{X}$, although the subclasses of A-groups and of metanilpotent groups are contained in $\mathfrak{X}$. An example described by Alperin (**1**) for other reasons would serve here, but it is perhaps of interest to establish directly for a different group G the fact that $G \notin \mathfrak{X}$.

EXAMPLE 5.6. *A Sylow tower group, of nilpotent length* 3, *need not belong to* $\mathfrak{X}$.

Construction. Let T be the split extension of a cyclic group of order 7 by a cyclic group of order 3, defined by

$$T = \{u, v\} \quad \text{and} \quad u^7 = v^3 = 1,\ u^v = u^2.$$

Then let G be the wreath product (according to regular representations) of a dihedral group of order 8 by T. Thus

$$G = HT,$$

where $H = \mathrm{Dr} \prod_{\substack{0 \leqslant r \leqslant 6 \\ 0 \leqslant s \leqslant 2}} \{x_{rs}, y_{rs}\}$, and

$$x_{rs}{}^4 = y_{rs}{}^2 = 1 = (x_{rs}y_{rs})^2,$$

$$x_{rs}{}^u = x_{r+4^s,s}, \quad y_{rs}{}^u = y_{r+4^s,s},$$

$$x_{rs}{}^v = x_{r,s+1}, \quad y_{rs}{}^v = y_{r,s+1},$$

for all r, s with $0 \leqslant r \leqslant 6$, $0 \leqslant s \leqslant 2$. (Here we adopt the convention that for any integers n, m such that $n \equiv r \pmod 7$ and $m \equiv s \pmod 3$, $x_{nm} = x_{rs}$ and $y_{nm} = y_{rs}$.)

Clearly G is a Sylow tower group, of nilpotent length 3; H is a normal Sylow subgroup of G, and G/H is metanilpotent.

Let D be the normalizer of the Sylow system of G determined by T, $H\{u\}$, and $H\{v\}$. The Sylow p-subgroup of D is denoted by D_p ($p = 2, 3, 7$). Since $H\{u\} \lhd G$, it follows that

$D_3 = \{v\}$, the Sylow 3-subgroup of G belonging to the Sylow system under consideration.

$D_7 = \{u\} \cap N_G(H\{v\}) = 1$, since $v^u = u^3 v \notin H\{v\}$.

$D_2 = H \cap N_G(T) = C_H(T)$, since $H \lhd G$ and $H \cap T = 1$. Thus

$$D = C_H(T) \times \{v\}.$$

Theorem 5.2 shows that $G \in \mathfrak{X}$ if and only if $D_2 = C_H(T) \lhd C_H(v)$.

Straightforward calculations show that

$$C_H(T) = \left\{\prod_{r,s} x_{rs}, \prod_{r,s} y_{rs}\right\}, \text{ a dihedral group of order 8;}$$

and that

$$C_H(v) = \{x_{00}x_{01}x_{02}, x_{10}x_{11}x_{12}, x_{20}x_{21}x_{22}, x_{30}x_{31}x_{32}, x_{40}x_{41}x_{42}, x_{50}x_{51}x_{52},$$
$$x_{60}x_{61}x_{62}, y_{00}y_{01}y_{02}, y_{10}y_{11}y_{12}, y_{20}y_{21}y_{22}, y_{30}y_{31}y_{32}, y_{40}y_{41}y_{42}, y_{50}y_{51}y_{52}, y_{60}y_{61}y_{62}\},$$

the direct product of 7 copies of a dihedral group of order 8.

Now

$$\begin{aligned}\left(\prod_{r,s} y_{rs}\right)^{x_{00}x_{01}x_{02}} &= y_{00}{}^{x_{00}} y_{01}{}^{x_{01}} y_{02}{}^{x_{02}} \left(\prod_{\substack{r,s\\ r\neq 0}} y_{rs}\right)\\ &= x_{00}{}^2 y_{00} x_{01}{}^2 y_{01} x_{02}{}^2 y_{02} \left(\prod_{\substack{r,s\\ r\neq 0}} y_{rs}\right)\\ &= (x_{00}x_{01}x_{02})^2 \left(\prod_{r,s} y_{rs}\right)\\ &\notin C_H(T).\end{aligned}$$

Therefore $C_H(T) \ntriangleleft C_H(v)$, and so $G \notin \mathfrak{X}$.

Example 5.6 may be used to show that $\mathfrak{X}$, although a formation (Theorem 3.4), is not a *saturated* formation. This is proved indirectly. Gaschütz and Lubeseder (**7**) have proved that a formation $\mathfrak{F}$ is saturated if and only if a finite group G belongs to $\mathfrak{F}$ when $G/\Phi(G)$ belongs to $\mathfrak{F}$ (where $\Phi(G)$ denotes the Frattini subgroup of G). We show, by induction on $|G|$, that if $\mathfrak{X}$ were saturated then any finite soluble group G of p-length

1 for every prime factor p of $|G|$ would belong to $\mathfrak{X}$. We may suppose that G is monolithic, and, assuming $\mathfrak{X}$ saturated, that $\Phi(G) = 1$. The unique minimal normal subgroup P of G is an abelian p-group, for some prime factor p of $|G|$. By the induction hypothesis, $G/P \in \mathfrak{X}$. There is a maximal subgroup M of G such that $M \not\geqslant P$, and it follows from this that $MP = G$ and $M \cap P = 1$. Then $C_G(P) = P$, and so P is the Fitting subgroup of G. But since G has by hypothesis p-length 1, P must also be the Sylow p-subgroup of G. Then Corollary 5.3 implies that $G \in \mathfrak{X}$, and this completes the induction argument.

Since Example 5.6 shows that the conclusion of this argument is false, we deduce that the assumption that $\mathfrak{X}$ is saturated is false.

6. Further investigation of the class $\mathfrak{X}$

We shall now consider more generally the problem of characterizing the class $\mathfrak{X}$. In view of 1.6, a necessary condition for a soluble group G to belong to $\mathfrak{X}$ is that a system normalizer D of G should have $N_G(D)$ as subnormalizer in G, or in other words that $D \lhd E$ whenever D is subnormal in a subgroup E of G. We ask whether this rather striking property is also sufficient to ensure that $G \in \mathfrak{X}$. Theorem 5.5 implies that this is true when G has p-length 1 for every prime factor p of $|G|$. It will be shown in this section that the property is always sufficient: thus we shall prove

THEOREM 6.1.† *Let G be a finite soluble group, and D a system normalizer of G. Then $G \in \mathfrak{X}$ if and only if D has $N_G(D)$ as subnormalizer in G.*

We begin by establishing a particular case, which will provide the means of applying an induction argument to prove the general result.

LEMMA 6.2. *Suppose that the finite soluble group G has an abelian normal subgroup A such that $G/A \in \mathfrak{X}$. Let D be a system normalizer of G, and suppose that D has $N_G(D)$ as subnormalizer in G. Then $G \in \mathfrak{X}$.*

Proof. Since DA/A is a system normalizer of G/A, DA/A is by hypothesis pronormal in G/A. Therefore DA is pronormal in G, so that for any x in G there exists an element y of $\{(DA), (DA)^x\}$ such that $(DA)^{xy} = DA$. Let $J = \{D, D^x\}$. Then $y \in JA$; and we may suppose without loss of generality that in fact $y \in J$.

Let $U = N_{DA}{}^{\infty}(D)$. Then, since D is subnormal in U and so by hypothesis $D \lhd U$, it follows that $U = N_{DA}(D)$. By Lemma 3.3, U is a Carter

† After this paper had been accepted for publication, I learned that Dr Alperin had also obtained this result.

subgroup of DA. Then U and U^{xy} are Carter subgroups of DA, and from this and the fact that $D \leqslant U$ it follows that there is an element z of A such that $U^{xyz} = U$. Put $t = xyz$.

Since $D \leqslant U \leqslant DA$,

$$U = D(A \cap U) = DN_A(D) = D(A \cap H),$$

where $H = N_G(D)$. Since $D \lhd H$ and $A \cap H \lhd H$, it follows that $U \lhd H$. Now $D^t \lhd U^t = U \lhd H$, so that D is subnormal in $H^{t^{-1}}$. This implies by hypothesis that $D \lhd H^{t^{-1}}$, and so, by definition of H, that $H^{t^{-1}} \leqslant H$. Then, by equality of orders, $H^{t^{-1}} = H$. Certainly D is contained in some Carter subgroup C of G. Since C is nilpotent, D is subnormal in C, and therefore $D \lhd C$. Hence $C \leqslant H$, and consequently H is abnormal in G. It follows then that $t \in H$, and hence that $D^t = D$. This yields $D^{xy} = D^{z^{-1}}$. Put $z^{-1} = a \in A$.

Let $K = \{D, D^a\} = \{D, D^{xy}\} \leqslant \{D, D^x, y\} = J$. It follows from

$$D \leqslant K \leqslant DA$$

that $K = D(K \cap A)$. Since A is abelian, $K \cap A \lhd A$. Hence

$$K^a = D^a(K \cap A) \leqslant K,$$

so that, by equality of orders, $K^a = K$.

Let $V = N_K{}^{\infty}(D)$. By the same argument as was used for U, it follows that $V = N_K(D)$. By Lemma 3.3, and since $K = D(K \cap A)$ and $K \cap A \lhd K$, V is a Carter subgroup of K. Then V and V^a are Carter subgroups of K, so that there is an element k of K such that $V^{ak} = V$. Moreover, $V \leqslant U = N_{DA}(D)$, since $K \leqslant DA$. But $U \lhd H$, and since U is nilpotent it follows that D^{ak} is subnormal in H. Therefore D is subnormal in $H^{(ak)^{-1}}$, and so $D \lhd H^{(ak)^{-1}}$. Then, arguing as before, we deduce that $H^{(ak)^{-1}} \leqslant H$; therefore $H^{(ak)^{-1}} = H$ and, since H is abnormal in G, $ak \in H$. Thus $D^{ak} = D$. Finally, $D^{xyk} = D^{ak} = D$ and $yk \in J$, since $K \leqslant J$. Therefore D is pronormal in G, and $G \in \mathfrak{X}$.

Corollary 6.3. *Let G be a finite soluble group. Then $G \in \mathfrak{X}$ if and only if, for each homomorphic image $\bar{G}$ of G, a system normalizer $\bar{D}$ of $\bar{G}$ has $N_{\bar{G}}(\bar{D})$ as subnormalizer in $\bar{G}$.*

Proof. The necessity of the condition for G to belong to $\mathfrak{X}$ follows from the fact that $\mathfrak{X}$ is Q-closed, and from 1.6. In order to prove the sufficiency, we use induction on $|G|$. G may be supposed non-trivial, and then G has a non-trivial abelian normal subgroup A. Since every homomorphic image of G/A is also a homomorphic image of G, the induction hypothesis implies that $G/A \in \mathfrak{X}$. Then Lemma 6.2 shows that $G \in \mathfrak{X}$.

In order to prove Theorem 6.1, it is enough, by Corollary 6.3, to show that the class of all finite soluble groups G for which a system normalizer D of G has $N_G(D)$ as subnormalizer in G is Q-closed. The method of establishing this fact was suggested by the referee, to whom my appreciation and thanks are given.

The key to the proof is the equivalence, pointed out by the referee, of the two properties described in the following lemma. In statement (b), z_0 denotes the invariant with the same designation introduced by Carter (**5**).

LEMMA 6.4. *Let G be a finite soluble group, and let D be a system normalizer of G. Then the following two statements are equivalent:*

(a) *D has $N_G(D)$ as subnormalizer in G;*

(b) $|N_G(D):D| = z_0(D)$.

Proof. This is contained in the proof of Theorem 14 of (**5**). Let $H = N_G(D)$. If D has H as subnormalizer in G, then by Theorem 14(i) of (**5**), $|H:D| = z_0(D)$.

Conversely, suppose that $|H:D| = z_0(D)$. Then, following in outline the latter part of the proof for Theorem 14(ii) of (**5**), we find that each Sylow system of G reducible into D is also reducible into H; and each Sylow system of G is reducible into only one conjugate in G of H. Hence D is contained in only one conjugate in G of H. Then it follows as in (**5**) that H is the subnormalizer in G of D.

Let X be any subgroup of a finite soluble group G. For any maximal chain $\mathscr{C}$ of subgroups joining X to G, $\beta(\mathscr{C})$ denotes the product of the indices of the normal links in $\mathscr{C}$. It is shown in § 2 of (**5**) that $z_0(X)$ may be characterized as the greatest value assumed by $\beta(\mathscr{C})$, as $\mathscr{C}$ ranges over all maximal chains of subgroups joining X to G. Then we may remark that if K and L are subgroups of G such that $X \leqslant L \lhd K \leqslant G$, then $z_0(X) \geqslant |K:L|$; for since K/L is soluble, a composition series of K/L corresponds to a maximal chain joining L to K in which every link is normal, and this can be extended to a maximal chain joining X to G.

We are now in a position to prove Theorem 6.1, since it is easy to show that property (b) of Lemma 6.4 characterizes a Q-closed class of groups.

Proof of 6.1. Let G be a finite soluble group and D a system normalizer of G. Suppose that D has H as subnormalizer in G, where $H = N_G(D)$. Then by Lemma 6.4, $|H:D| = z_0(D)$. Suppose that $K \lhd G$. Then since $DK \lhd HK$, the remark above shows that

$$z_0(DK) \geqslant |HK:DK| = |H:H \cap (DK)|. \tag{1}$$

Any maximal chain joining D to DK may be extended by any maximal chain joining DK to G to form a maximal chain joining D to G. Since $D \lhd H \cap (DK)$, this implies that

$$z_0(D) \geqslant z_0(DK) | H \cap (DK) : D |;$$

that is, by (1),

$$z_0(D) \geqslant z_0(DK) | H \cap (DK) : D | \geqslant | H : D |. \tag{2}$$

But since $z_0(D) = | H : D |$, equality must hold in (2), and therefore

$$z_0(DK) = | HK : DK |.$$

Hence

$$z_0(DK/K) = z_0(DK) = | HK : DK | = | HK/K : DK/K |.$$

Since $DK/K \lhd HK/K$, it follows from this that $HK/K = N_{G/K}(DK/K)$. Now DK/K is a system normalizer of G/K, and so, by Lemma 6.4, DK/K has $N_{G/K}(DK/K)$ as subnormalizer in G/K. Since K is an arbitrary normal subgroup of G, Corollary 6.3 shows that $G \in \mathfrak{X}$.

We recall Carter's results on groups of nilpotent length at most 3 ((**5**) or (**2**)). If G is a soluble group and $l(G) \leqslant 3$, then each system normalizer D of G is contained in a single Carter subgroup C of G, and C is the subnormalizer of D in G. This leads to a simple criterion for such a group to belong to $\mathfrak{X}$.

COROLLARY 6.5. *Suppose that G is a finite soluble group with $l(G) \leqslant 3$. Let D be a system normalizer of G, and C the unique Carter subgroup of G containing D. Then $G \in \mathfrak{X}$ if and only if $D \lhd C$.*

Proof. The necessity of the condition for G to belong to $\mathfrak{X}$ follows from 1.5. In order to prove sufficiency, we observe that since C is the subnormalizer of D in G, the condition $D \lhd C$ implies that $C = N_G(D)$, and therefore that $N_G(D)$ is the subnormalizer of D in G. Then by Theorem 6.1, $G \in \mathfrak{X}$.

In view of the simplicity of the condition in Corollary 6.5, one might be tempted to hope that an analogous result would be generally valid, namely that if G were a finite soluble group, D a system normalizer, and C a Carter subgroup of G, the condition $D \lhd C$ would imply that $G \in \mathfrak{X}$. That this is false, even when G has nilpotent length 4, is shown by the following example, for which I am indebted to Dr J. L. Alperin. I am grateful to him for his permission to include it here.

EXAMPLE 6.6. *A finite soluble group G, of nilpotent length 4, may have a system normalizer D and Carter subgroup C such that $D \lhd C$, yet $G \notin \mathfrak{X}$.*

Construction. Let H be a split extension of an elementary abelian group $\{x\} \times \{y\}$ of order 5^2 by a symmetric group $\{u, v\}$ of degree 3, defined by $x^5 = y^5 = u^3 = v^2 = 1$, $xy = yx$, $u^v = u^{-1}$, $x^u = y$, $y^u = x^{-1}y^{-1}$, $x^v = x^{-1}$, $y^v = xy$. Then $|H| = 2.3.5^2$. $D = \{v\}$ is a system normalizer of H (since, by covering and avoidance properties, the system normalizers of H have order 2, and are thus the Sylow 2-subgroups of H). There is a (unique) Carter subgroup C of H containing D, and certainly $C \leqslant \{x, y\}D$. Since C is a maximal nilpotent subgroup of H, we find then that $C = \{xy^2\} \times \{v\}$. Thus $D \lhd C$.

Now H has a faithful representation of degree 3 over the Galois field $\mathrm{GF}(2^4)$, by means of the mapping

$$x \to \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad y \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^{-1} \end{pmatrix},$$

$$u \to \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad v \to \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where λ is a primitive fifth root of 1 in $GF(2^4)$; such roots exist since 5 is a divisor of $2^4 - 1$. It follows that H has also a faithful representation of degree 12 over $\mathrm{GF}(2)$. Therefore, by means of this representation, we may form a split extension G of an elementary abelian group A of order 2^{12} by H. The element xy^2 of C, which is represented over $\mathrm{GF}(2^4)$ by

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^{-2} \end{pmatrix},$$

centralizes no non-trivial element of A, so that $C_A(xy^2) = 1$. If $g \in N_G(C)$ then $g \in N_G(\{xy^2\})$. We may express g in the form $g = ah$, where $a \in A$, $h \in H$, and then

$$\{xy^2\}^a = \{xy^2\}^{h^{-1}} \leqslant (A\{xy^2\}) \cap H = \{xy^2\}.$$

Hence $a \in N_A(\{xy^2\}) = C_A(xy^2)$, since $A \lhd G$ and $A \cap H = 1$. Therefore $a = 1$, and so $N_G(C) = N_H(C) = C$. Thus C is a Carter subgroup of G; and it follows that D is a system normalizer of G. Moreover $D \lhd C$.

However, AD is a 2-group, and therefore D is subnormal in AD. If D were normal in AD, then since $A \lhd AD$ and $A \cap D = 1$, D would centralize A; but this is false: the element v does not centralize A. Therefore by 1.5, D is not pronormal in AD, and so $G \notin \mathfrak{X}$.

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Newcastle upon Tyne

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A NATURAL SETTING FOR THE EXTENSIONS OF A GROUP WITH TRIVIAL CENTRE BY AN ARBITRARY GROUP

by John S. Rose

Let A and H be groups. The intention of the present note is to point out that if A has trivial centre, then any extension of A by H is equivalent (in the sense of extension theory) to one determined in a natural way by a suitable subgroup of Aut $A \times H$. This fact is already implicit in an early paper of R. Baer [1][1], but the aim here is to provide a more explicit formulation; and to deduce that the non-equivalent extensions of a group A with trivial centre by an arbitrary group H stand in one-to-one correspondence with the distinct homomorphisms of H into the group Aut A/Inn A of automorphism classes of A. This latter result is obtained in the treatment of Kurosh [5, p. 148] as a corollary of some cohomological theorems of S. Eilenberg and S. MacLane [3]. The proof offered here is an entirely elementary application of the fact that it is possible to work within Aut $A \times H$.

The notation and terminology used are largely standard. For an arbitrary group A, Aut A denotes the group of all automorphisms of A and Inn A the group of all inner automorphisms of A. We shall denote by $\mathfrak{A}(A)$ the group Aut A/Inn A of automorphism classes of A. If B is a subgroup of A, $C_A(B)$ denotes the centralizer of B in A. Then $C_A(A) = Z(A)$, the centre of A. For an arbitrary element a of A, the inner automorphism of A induced by a is denoted by τ_a; this notation is relative to the group A, which is here presumed to be fixed. The groups $A/Z(A)$ and Inn A may be identified in the natural way by identifying, for each a in A, the elements $aZ(A)$ and τ_a: this identification will be made. An automorphism α of A is called a central automorphism if, for every a in A, $(a\alpha)\, a^{-1} \in Z(A)$. It is easy to show that the set of all central automorphisms of A forms a subgroup of Aut A which is in fact precisely $C_{\mathrm{Aut}A}$ (Inn A): see Zassenhaus [6, p. 52].

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1) See also H. Fitting [4, § 21].

Let A and H be arbitrary groups. An *extension* of A by H is a pair (G, φ) consisting of a group G, containing A as a normal subgroup, and a homomorphism φ of G onto H such that $\operatorname{Ker} \varphi = A$. (Reference to the particular homomorphism φ involved in an extension is often omitted, for instance in KUROSH [5, Chapter XII], but φ is tacitly assumed to be specified in the development of the theory.) Two extensions (G, φ) and (G^*, φ^*) of A by H are said to be *equivalent* if there is an isomorphism Θ of G onto G^* mapping A identically onto itself and such that $\Theta\varphi^* = \varphi$.

Suppose that (G, φ) is an extension of A by H, and that B is a characteristic subgroup of A. Then B is a normal subgroup of G, and (G, φ) induces naturally an extension $(G/B, \bar{\varphi})$ of A/B by H: $\bar{\varphi}$ is defined by

$$(gB)\,\bar{\varphi} = g\varphi\,, \quad \text{for any } g \text{ in } G;$$

this is well defined since $B \leqq A = \operatorname{Ker} \varphi$. We shall be concerned with the special case in which $B = Z(A)$.

Let $\bar{A} = A/Z(A)$. It is possible, for arbitrary groups A and H, to construct extensions of $\bar{A}$ by H rather transparently by means of suitable subgroups of $\operatorname{Aut} A \times H$ (the external direct product). $\operatorname{Aut} A$ is identified with a subgroup of this direct product in the obvious way by identification of α with $(\alpha, 1)$, for each α in $\operatorname{Aut} A$. Then $\bar{A}$, which is identified with $\operatorname{Inn} A$, is also identified with a subgroup of $\operatorname{Aut} A \times H$. Let π denote the projection homomorphism of $\operatorname{Aut} A \times H$ onto H:

$$(\alpha, h)\,\pi = h, \quad \text{for any } \alpha \text{ in } \operatorname{Aut} A \text{ and } h \text{ in } H.$$

Then any subgroup Q of $\operatorname{Aut} A \times H$ such that $Q \cap \operatorname{Aut} A = \bar{A}$ and $Q\pi = H$ determines an extension (Q, π_0) of $\bar{A}$ by H, where π_0 is simply the restriction of π to Q: for π_0 is a homomorphism of Q onto H, since $Q\pi = H$, and $\operatorname{Ker} \pi_0 = Q \cap \operatorname{Ker} \pi = Q \cap \operatorname{Aut} A = \bar{A}$. For convenience, we introduce a term for such an extension: we shall call it a *sited extension* of $\bar{A}$ by H.

We shall prove the

THEOREM. *Let A and H be arbitrary groups. Suppose that (G, φ) is an extension of A by H, and let $(\bar{G}, \bar{\varphi})$ be the induced extension of $\bar{A}$ by H, where $\bar{G} = G/Z(A)$, $\bar{A} = A/Z(A)$. Then $(\bar{G}, \bar{\varphi})$ is equivalent to a sited extension of $\bar{A}$ by H. Moreover, if the only central automorphisms of A are* *inner automorphisms, then sited extensions of $\bar{A}$ by H corresponding to distinct subgroups of* Aut $A \times H$ *are non-equivalent.*

Proof. For any element g of G, let σ_g denote the restriction to A of the inner automorphism of G induced by g. (Thus $\sigma_a = \tau_a$, for each a in A.) We define a map $\psi \colon G \to \operatorname{Aut} A \times H$ by

$$g\psi = (\sigma_g, g\varphi)\,, \qquad \text{for every } g \text{ in } G.$$

Clearly ψ is a homomorphism, and

$$\begin{aligned} \operatorname{Ker} \psi &= \{ g \in G \mid g^{-1} ag = a \text{ for all } a \text{ in } A \} \cap \operatorname{Ker} \varphi \\ &= C_G(A) \cap A \\ &= Z(A)\,. \end{aligned}$$

Then ψ induces naturally an isomorphism $\bar{\psi}$ of $\bar{G}$ onto a subgroup Q of Aut $A \times H$; and

$$\begin{aligned} Q \cap \operatorname{Aut} A &= \{ (\sigma_g, g\varphi) \mid g \in G, g\varphi = 1 \} \\ &= \{ (\sigma_g, 1) \mid g \in \operatorname{Ker} \varphi \} \\ &= \bar{A}\,, \\ Q\pi &= \{ g\varphi \mid g \in G \} \\ &= \operatorname{Im} \varphi \\ &= H\,. \end{aligned}$$

Hence Q determines a sited extension (Q, π_0) of $\bar{A}$ by H, where π_0 is the restriction of π to Q.

We show that $(\bar{G}, \bar{\varphi})$ is equivalent to (Q, π_0). For this purpose we can use $\bar{\psi}$, which is an isomorphism of $\bar{G}$ onto Q. For any element g of G, let $\bar{g} = g\, Z(A)$. Then

$$\bar{g}\,(\bar{\psi}\pi_0) = (g\psi)\,\pi_0 = (\sigma_g, g\varphi)\,\pi = g\varphi = \bar{g}\bar{\varphi}\,,$$

so that

$$\bar{\psi}\pi_0 = \bar{\varphi}\,.$$

Also, for any a in A,

$$\bar{a}\bar{\psi} = a\psi = (\sigma_a, a\varphi) = (\tau_a, 1) = \bar{a}\,,$$

by identification, so that $\bar{\psi}$ maps $\bar{A}$ identically onto itself. This establishes the equivalence of $(\bar{G}, \bar{\varphi})$ and (Q, π_0).

Now assume that the only central automorphisms of A are inner, that is that $C_{\mathrm{Aut}A}(\bar{A}) \leqslant \bar{A}$. Suppose that Q, Q^* are subgroups of Aut $A \times H$ such that $Q \cap \mathrm{Aut}\, A = \bar{A} = Q^* \cap \mathrm{Aut}\, A$ and $Q\pi = H = Q^*\pi$, so that Q, Q^* determine sited extensions (Q, π_0), (Q^*, π_0^*) of $\bar{A}$ by H. Suppose that these extensions are equivalent. Then there is an isomorphism Θ of Q onto Q^* mapping $\bar{A}$ identically onto itself and such that $\Theta\pi_0^* = \pi_0$.

It is unnecessary to make this choice. We may argue directly with any $(\alpha, h) \in Q$. Then the paragraph at the top of p. 171 is redundant.

For each h in H, we choose α_h in Aut A such that $(\alpha_h, h) \in Q$: this is possible since $Q\pi = H$. (In general h does not determine a unique such element α_h, but we make a choice of one element for each h.) Let

$$(\alpha_h, h)\,\Theta = (\alpha_h^*, h^*), \qquad \text{with } \alpha_h^* \text{ in Aut } A \text{ and } h^* \text{ in } H.$$

Since $\bar{A}$ is a normal subgroup of Aut A,

$$\alpha_h^{-1}\bar{a}\ \alpha_h \in \bar{A} \quad \text{for any } \bar{a} \text{ in } \bar{A}\,,$$

and therefore

$$(\alpha_h^{-1}\bar{a}\,\alpha_h)\ \ \Theta = \alpha_h^{-1}\,\bar{a}\,\alpha_h\,. \tag{1}$$

Now (by identification)

$$\alpha_h^{-1}\bar{a}\,\alpha_h = (\alpha_h, h)^{-1}\bar{a}\,(\alpha_h, h)\,. \tag{2}$$

Since (α_h, h) and $\bar{a}$ both belong to Q, (1) and (2) give

$$\begin{aligned}\alpha_h^{-1}\bar{a}\,\alpha_h &= \big((\alpha_h, h)\,\Theta\big)^{-1}(\bar{a}\Theta)\big((\alpha_h, h)\,\Theta\big)\\ &= (\alpha_h^*, h^*)^{-1}\,\bar{a}\,(\alpha_h^*, h^*)\,,\end{aligned}$$

that is

$$\alpha_h^{-1}\bar{a}\,\alpha_h = (\alpha_h^*)^{-1}\bar{a}\ \alpha_h^*\,. \tag{3}$$

Hence $\alpha_h^*\alpha_h^{-1} \in C_{\mathrm{Aut}\,A}(\bar{A}) \leqslant \bar{A}$, by hypothesis. Thus for each h in H, there is an element η_h in $\bar{A}$ such that

$$\alpha_h^* = \eta_h\,\alpha_h\,. \tag{4}$$

Also

$$h^* = (\alpha_h, h)\,\Theta\pi_0^* = (\alpha_h, h)\,\pi_0 = h\,,$$

so that

$$(\alpha_h, h)\,\Theta = (\alpha_h^*, h)\,,$$

that is

$$(\alpha_h, h)\,\Theta = \eta_h\,(\alpha_h, h)\,. \tag{5}$$

Now we consider an arbitrary element of Q, say (α, h) with α in Aut A and h in H. Since also $(\alpha_h, h) \in Q$ and $Q \cap \text{Aut } A = \bar{A}$, there is an element $\bar{a}$ in $\bar{A}$ such that

$$(\alpha, h) = \bar{a}(\alpha_h, h).$$

Then

$$\begin{aligned}(\alpha, h)\,\Theta &= (\bar{a}\Theta)\big((\alpha_h, h)\,\Theta\big) \\ &= \bar{a}\,\eta_h(\alpha_h, h), \qquad \text{by (5).}\end{aligned}$$

Since $\bar{A} \leqq Q$, this shows that $(\alpha, h)\,\Theta \in Q$. Hence $Q^* = Q\Theta \leqq Q$. Similarly $Q \leqq Q^*$. Therefore $Q = Q^*$. This complete the proof.

We observe now that the distinct subgroups of Aut $A \times H$ determining sited extensions of $\bar{A}$ by H stand in one-to-one correspondence with the distinct homomorphisms of H into $\mathfrak{A}(A)$. To see this, suppose first that Q is a subgroup of Aut $A \times H$ determining a sited extension of $\bar{A}$ by H, that is such that $Q \cap \text{Aut } A = \bar{A}$ and $Q\pi = H$. Then Q determines a homomorphism $\lambda_Q \colon H \to \mathfrak{A}(A)$ as follows:

$$\text{for any } h \text{ in } H,\ h\lambda_Q = \alpha\bar{A} \text{ if and only if } (\alpha, h) \in Q,$$

where $\alpha \in \text{Aut } A$.

Since $Q \cap \text{Aut } A = \bar{A}$, λ_Q is well defined by this rule, and is defined on the whole of H since $Q\pi = H$. Conversely, suppose that λ is a homomorphism of H into $\mathfrak{A}(A)$. Then λ determines a subgroup Q of Aut $A \times H$, defined as

$$Q = \{\, (\alpha, h) \mid \alpha \in \text{Aut } A,\ h \in H \text{ and } h\lambda = \alpha\bar{A} \,\},$$

and it is clear that then $Q \cap \text{Aut } A = \bar{A}$ and $Q\pi = H$, so that Q determines a sited extension of $\bar{A}$ by H. Furthermore, $\lambda_Q = \lambda$. Finally, distinct homomorphisms of H into $\mathfrak{A}(A)$ evidently determine distinct subgroups of Aut $A \times H$, and so the correspondence between homomorphisms and subgroups is one-to-one.

If A is a group with trivial centre, then A is naturally identified with $\bar{A}$ and the Theorem shows that any extension of A by H is equivalent to a sited extension of A by H. Moreover, the only central automorphism of A is the identity automorphism, so that we obtain

Corollary 1. *Let A be a group with trivial centre and H an arbitrary group. Then every extension of A by H is equivalent to a sited extension* *of A by H. The non-equivalent extensions of A by H stand in one-to-one correspondence with the distinct homomorphisms of H into $\mathfrak{A}(A)$.*

If the only homomorphism of H into $\mathfrak{A}(A)$ is the trivial homomorphism, then the only sited extension of $\bar{A}$ by H is $(\bar{A} \times H, \pi)$, where π denotes the projection map of $\bar{A} \times H$ onto H. Thus in particular we have

COROLLARY 2. *Let A be a group with trivial centre and H a group. Then (up to equivalence) the only extension of A by H is $(A \times H, \pi)$, where π denotes the projection map of $A \times H$ onto H, in any of the following cases:*

(i) $\mathfrak{A}(A)$ *is trivial.*

(ii) $\mathfrak{A}(A)$ *is soluble and* H *is perfect.*

(iii) $\mathfrak{A}(A)$ *is a ϖ-group and H is a ϖ'-group, where ϖ is a set of prime numbers and ϖ' the set of all prime numbers not belonging to ϖ.*

(iv) *H is simple and cannot be embedded in $\mathfrak{A}(A)$.*

Here (i) is the well known case of a *complete* group A.

According to a celebrated conjecture of O. SCHREIER, $\mathfrak{A}(E)$ ought to be soluble for any finite non-abelian simple group E. SCHREIER's Conjecture is valid for every known finite non-abelian simple group. Thus (ii) applies if A is any known finite non-abelian simple group.

Another result can be derived from (ii) and a Lemma due to H. FITTING [4, Satz 12], which may be expressed as follows.

LEMMA. *Let E be a finite non-abelian simple group. Then, if* n *is a positive integer and D is the direct product of n copies of E,* Aut *D is isomorphic to the wreath product of* Aut *E by the symmetric group of degree n, formed according to the natural representation.*

A group is called *completely reducible* if it can be decomposed as a direct product of a finite number of simple groups (KUROSH [5, p. 203]). An easy inductive proof, using (ii) and the Lemma, establishes

COROLLARY 3. *Let G be a non-trivial finite group. Associated with G there is a set of non-isomorphic simple groups $E_1, \ldots, E_k$ and a set of positive integers $n_1, \ldots, n_k$ such that every composition series of G has precisely n_i composition factors isomorphic to E_i $(i = 1, \ldots, k)$ and no others. If every E_i is non-abelian and satisfies* SCHREIER's *Conjecture, and if every $n_i < 5$, then G is completely reducible.*

This is a particular case of a recent result of R. BERCOV [2].

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University
of Newcastle upon Tyne,
England.

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Nilpotent Subgroups of Finite Soluble Groups

JOHN S. ROSE

The general problem, with a particular instance of which the present paper is concerned, is to obtain a description of the local structure of a group from information about the global structure. The aspect of local structure investigated here is the embedding of subgroups, especially of nilpotent subgroups in finite soluble groups. A classification of embeddings of subgroups in finite groups by means of an arithmetic function called abnormal depth was proposed in [6]. Let H be a subgroup of a finite group G. Then $a(G:H)$, the *abnormal depth* of H in G, is the least number of abnormal links appearing in any *balanced* chain of subgroups connecting H to G, that is a chain for which each link is either normal or abnormal. Thus $a(G:H)=0$ if and only if H is subnormal in G; and $a(G:P)\leqq 1$ for every subgroup P of G of prime power order. It was shown in [6] that if H is a nilpotent subgroup of a finite soluble group G, of nilpotent length n, then $a(G:H)\leqq n-1$. Here in § 1 we examine in greater detail the easiest non-trivial case, in which $n=2$, and then in § 2 prove certain supplementary results for $n=3$ and $n=4$. Some simple wreath product properties are established in § 3 and used in § 4 for the construction of examples showing that the embedding results obtained cannot be improved in various obvious ways.

Notation and terminology follow common usage. If $\mathfrak{X}$ and $\mathfrak{Y}$ are classes of groups, then $\mathfrak{X}\mathfrak{Y}$ denotes the class of all groups G having a normal subgroup X such that $X\in\mathfrak{X}$ and $G/X\in\mathfrak{Y}$. This defines a composition of classes of groups which in general is not associative. However, we shall deal only with classes of which the composition is associative, and we may therefore omit brackets from products of more than two classes. Since we shall be concerned exclusively with finite groups, we take $\mathfrak{N}$ to denote the class of finite nilpotent groups and $\mathfrak{A}$ the class of finite abelian groups. Then for any positive integer n, $\mathfrak{N}^n$ is the class of finite soluble groups of nilpotent lengths $\leqq n$; and $\mathfrak{A}^n$ is the class of finite soluble groups of derived lengths $\leqq n$.

Henceforth the term group is understood to mean finite group. Then any group G has a unique smallest normal subgroup L such that G/L is nilpotent: G/L is called the $\mathfrak{N}$-*residual* of G. If H is any subgroup of G, then there is a unique smallest normal subgroup of G containing H, called the *normal closure* of H in G and denoted by H^G; and a unique smallest subnormal subgroup of G containing H, called the *subnormal closure* of H in G and (following Wielandt [8]) denoted by $H^{\cdot\cdot G}$. If $H^G=G$, we shall say that H is *contranormal* in G. Then, for any subgroup H of G, it is clear that H is contranormal in $H^{\cdot\cdot G}$. (This is to be compared with the fact that the hypernormalizer $N_G^\infty(H)$ of H in G is self-normalizing in G.) An abnormal subgroup is both self-normalizing and contranormal; though a subgroup may be both self-normalizing and contranormal without being abnormal, as the existence of subgroups of abnormal depth >1 shows. A contranormal subgroup of G necessarily covers the $\mathfrak{N}$-residual of G. If G is soluble, the converse is also true: a subgroup which covers the $\mathfrak{N}$-residual of G is contranormal in G.

The present enquiry directs attention to contranormal nilpotent subgroups. It is by now well known that a soluble group G possesses self-normalizing nilpotent subgroups and that all such subgroups are conjugate in G (Carter [3]): these are the Carter subgroups of G. They are in fact abnormal in G, therefore in particular contranormal in G. But in general G has contranormal nilpotent subgroups other than its Carter subgroups. For instance, any system normalizer of G is a contranormal nilpotent subgroup of G; it is contained in, but in general does not coincide with, a Carter subgroup of G. We recall, however, that if $G \in \mathfrak{N}^2$ then the class of system normalizers of G is the same as the class of Carter subgroups of G (Carter [1, Theorem 5.6]). In various situations, we shall ask the question: is a contranormal nilpotent subgroup necessarily contained in a Carter subgroup?

If H is any subgroup of G and X any non-empty subset of G, $N_H(X)$ denotes the normalizer of X in H and $C_H(X)$ the centralizer of X in H. Thus $N_H(X) = H \cap N_G(X)$ and $C_H(X) = H \cap C_G(X)$. Also $C_G(G) = Z(G)$, the centre of G. The derived group of G is denoted by G'. The subgroup of G generated by X is denoted by $\langle X \rangle$, and if Y is another subset of G, the notation $\langle X, Y \rangle$ is used in place of $\langle X \cup Y \rangle$. If x and y are arbitrary elements of G, $H^x = x^{-1}Hx$ and $y^x = x^{-1}yx$.

The symbol p always denotes a prime number. If H is a nilpotent group, then H_p denotes the unique Sylow p-subgroup of H and H^p the unique p-complement of H.

§ 1. Nilpotent Subgroups of $\mathfrak{N}^2$-groups

The basic lemma for the development of §§ 1 and 2 is the following, proved in [6].

Lemma 1.1. *Suppose that $G = HK$, with $K \trianglelefteq G$ and H, K nilpotent. Then $N_G^\infty(H)$ is a Carter subgroup of G.*

Since a contranormal subgroup of a group G covers the $\mathfrak{N}$-residual of G, this has the

Corollary 1.2. *If $G \in \mathfrak{N}^2$ and H is a contranormal nilpotent subgroup of G, then H is contained in a Carter subgroup of G.*

If $G \in \mathfrak{A}\mathfrak{N}$, more precise information is obtainable.

Corollary 1.3. *If $G \in \mathfrak{A}\mathfrak{N}$, then the only contranormal nilpotent subgroups of G are the Carter subgroups of G.*

Proof. Let H be a contranormal nilpotent subgroup of G, and let G/L be the $\mathfrak{N}$-residual of G. Then $HL = G$, and by 1.1, $H \leqq J$, a Carter subgroup of G. Since $G \in \mathfrak{N}^2$, J is also a system normalizer of G, and therefore, since $G \in \mathfrak{A}\mathfrak{N}$,

$J \cap L = 1$ (Carter [2, Theorem 2]). Moreover, $JL = G$. Thus $|H|$ divides $|J|$, $|J| = |G/L|$ and $|G/L|$ divides $|H|$. Hence $|H| = |J|$, and so $H = J$.

From 1.2 and 1.3 we deduce

Corollary 1.4. (i) *If* $G \in \mathfrak{A}\mathfrak{N}$ *and* H *is any nilpotent subgroup of* G, *then* H *is abnormal in* $H^{\cdot\cdot G}$.

(ii) *If* $G \in \mathfrak{N}^2$ *and* H *is a maximal nilpotent subgroup of* G, *then* H *is abnormal in* $H^{\cdot\cdot G}$.

Proof. In any case, H is contranormal in $H^{\cdot\cdot G}$.

For (i), apply 1.3, since $H^{\cdot\cdot G} \in \mathfrak{A}\mathfrak{N}$. For (ii), apply 1.2: H is contained in a Carter subgroup J of $H^{\cdot\cdot G}$, and then the maximality of H implies that $H = J$.

1.4 (i) yields, by a simple induction argument, a more general result:

Corollary 1.5. *Suppose that* $G \in \mathfrak{A}\mathfrak{N}$. *If* H *is any subgroup of* G, *then* H *is abnormal in* $H^{\cdot\cdot G}$. *In particular, any contranormal subgroup of* G *is abnormal in* G.

Proof. This is simply a repetition of the proof of Theorem 2 of [6], with routine modifications.

1.4 may be used to obtain some information about the structure of the subnormal closure of any subgroup of prime power order in an $\mathfrak{A}\mathfrak{N}$-group. (For a general discussion of subnormal closures of p-subgroups, see Wielandt [8, § 3].)

Proposition 1.6. *Suppose that* $G \in \mathfrak{A}\mathfrak{N}$. *If* P *is any* p-*subgroup of* G, *then* $P^{\cdot\cdot G}$ *has an abelian normal* p-*complement* R; $P^{\cdot\cdot G} = PR$ *and* $P \cap R = 1$.

Proof. Let $K = P^{\cdot\cdot G}$. By 1.4 (i), P is a Carter subgroup of K. It follows, since P is a p-subgroup, that P is a Sylow p-subgroup of K; and also, because $K \in \mathfrak{N}^2$, that P is a system normalizer of K. Therefore (Hall [5, Theorem 3.3])

$$P = P \cap N_K(R),$$

for some p-complement R of K. This implies that $R \trianglelefteq K$. Now P covers the $\mathfrak{N}$-residual of K, and so, since P is a p-group, the $\mathfrak{N}$-residual of K is a p-group. It follows that K/R is the $\mathfrak{N}$-residual of K. But $K \in \mathfrak{A}\mathfrak{N}$, and therefore R must be abelian.

Remarks. 1. In general, a group in $\mathfrak{N}^2$ has contranormal nilpotent subgroups other than its Carter subgroups. For instance, in the non-nilpotent split extension X of a quaternion group $Q = \langle j, k \rangle$ by a cyclic group $Y = \langle y \rangle$ of order 3, defined by the relations $j^4 = k^4 = 1$, $j^2 = k^2$, $kj = j^3 k$, $y^3 = 1$, $j^y = k$, $k^y = jk$, all proper subgroups of X are nilpotent. The Carter subgroups are thus maximal subgroups, in fact cyclic of order 6. They are the normalizers of the Sylow 3-subgroups of X, which are themselves contranormal in X because the $\mathfrak{N}$-residual of X has order 3.

2. We shall see in Examples 2 and 3 of § 4 that a contranormal nilpotent subgroup H of a group G in $\mathfrak{N}^3$ need not be contained in a Carter subgroup of G; although if $G \in \mathfrak{N}\mathfrak{A}\mathfrak{N}$, Proposition 2.1 will show that H is contained in a Carter subgroup of G.

3. By 1.5, if G is an $\mathfrak{A}\mathfrak{N}$-group, contranormal subgroups of G are necessarily abnormal in G; on the other hand, self-normalizing subgroups of G need not be abnormal. Indeed, if $\mathfrak{X}$ is any class of soluble groups, closed under the operations of forming quotients and direct squares, with the property that whenever H is a self-normalizing $\mathfrak{X}$-subgroup of an $\mathfrak{X}$-group G it follows that H is abnormal in G, then $\mathfrak{X} \subseteq \mathfrak{N}$. To establish this, it is enough to show that any non-trivial $\mathfrak{X}$-group has a non-trivial centre. Suppose to the contrary that there is a group $G \in \mathfrak{X}$ with $G \neq 1$, but $Z(G) = 1$. Then by hypothesis, $G^* = G \times G \in \mathfrak{X}$. Let $\hat{G}$ denote the diagonal subgroup of G^*. Since $\hat{G} \cong G$, $\hat{G} \in \mathfrak{X}$, and since $Z(G) = 1$, $N_{G^*}(\hat{G}) = \hat{G}$ ([6, Lemma 2]). Therefore, by hypothesis, $\hat{G}$ is abnormal in G^*. But since G is soluble and $G \neq 1$, $G' < G$, and so $\hat{G}$ is not contranormal in G^*, a fortiori not abnormal in G^*. We conclude from this contradiction that there is no such group G in $\mathfrak{X}$, and therefore that every non-trivial $\mathfrak{X}$-group has a non-trivial centre.

It follows from 1.4 (ii) that if $G \in \mathfrak{N}^2$ and H is any nilpotent subgroup of G, then there is a chain of subgroups

$$H \leqq J \leqq K \leqq G,$$

with H subnormal in J, J abnormal in K and K subnormal in G. (This is the case $n = 2$ of Theorem 1 in [6].) From 1.4 (i), we know also that if $G \in \mathfrak{A}\mathfrak{N}$, we can always choose $J = H$. We shall examine conditions under which we can choose $K = G$ (with in general $H \neq J$). For this it is natural to confine attention in the first place to maximal nilpotent subgroups H of G. We shall prove the equivalence of several conditions on a maximal nilpotent subgroup of a group in $\mathfrak{N}^2$, of which one is the existence of a chain of the kind under consideration.

For this purpose, it is convenient to make use of simple properties of pronormal subgroups. A subgroup H is said to be *pronormal* in a group G if, for any elements x and y of G, H^x and H^y are conjugate in $\langle H^x, H^y \rangle$: see [7, §1]. It is easy to show that if H is pronormal in G, then $N_G(H)$ is abnormal in G, and that H is both pronormal and subnormal in G if and only if $H \trianglelefteq G$. We note another property: if H is pronormal in G, then H is contranormal in H^G. To prove this let $K = H^{\cdot\cdot G}$. Then, for any x in G, $K^x = (H^x)^{\cdot\cdot G}$. Since H is pronormal in G, there is an element y in $\langle H, H^x \rangle$ such that $H^{xy} = H$. Then $K^{xy} = K$; and since $\langle H, H^x \rangle \leqq \langle K, K^x \rangle$, it follows that K is pronormal in G. But since K is also subnormal in G, this implies that $K \trianglelefteq G$. Hence $K = H^G$. We state the two properties which will be needed as

Lemma 1.7. *Suppose that H is a pronormal subgroup of a group G. Then $N_G(H)$ is abnormal in G, and H is contranormal in H^G.*

Proposition 1.8. *Suppose that $G \in \mathfrak{N}^2$ and let H be a maximal nilpotent subgroup of G. Then the following statements are equivalent:*

(i) *There is a chain of subgroups $H \leqq J \leqq G$, with H subnormal in J and J abnormal in G.*

(ii) *$N_G(H)$ is abnormal in G.*

(iii) *$N_G(H)$ is contranormal in G.*

(iv) *H is pronormal in G.*

(v) *H is contranormal in H^G.*

(vi) *H is abnormal in H^G.*

It is well known that if H is a maximal nilpotent subgroup of an arbitrary group G, then $N_G(H)$ is self-normalizing in G; cf. (ii) and (iii) above. Recall also 1.4 (ii).

Proof. (i)$\Rightarrow$(ii) H is a subnormal nilpotent subgroup of J, and therefore H is contained in the Fitting subgroup of J. Then by the maximality of H, H must coincide with the Fitting subgroup of J. Therefore $N_G(H) \geqq J$, and so $N_G(H)$ is abnormal in G.

(ii)$\Rightarrow$(iii) This is immediate.

(iii)$\Rightarrow$(iv) Let G/L be the $\mathfrak{N}$-residual of G. Then by hypothesis, $N_G(H)\,L = G$, so that any element x in G may be expressed in the form $x = u\,v$, with u in $N_G(H)$ and v in L. This gives $H^x = H^{uv} = H^v$. Because L is nilpotent and H is a maximal nilpotent subgroup of HL, 1.1 shows that H is a Carter subgroup of HL. Thus H is abnormal in HL, and so $v \in \langle H, H^v \rangle = \langle H, H^x \rangle$. Hence H is pronormal in G.

(iv)$\Rightarrow$(v) By 1.7.

(v)$\Rightarrow$(vi) The hypothesis implies that $H^{\cdot\cdot G} = H^G$, and then the conclusion follows by 1.4 (ii).

(vi)$\Rightarrow$(iv) For any x in G, $\langle H, H^x \rangle \leqq H^G$, and so the hypothesis that H is abnormal in H^G implies that H and H^x are Carter subgroups of $\langle H, H^x \rangle$. Therefore H and H^x are conjugate in $\langle H, H^x \rangle$. Hence H is pronormal in G.

(iv)$\Rightarrow$(ii)$\Rightarrow$(i) By 1.7.

Proposition 1.9. *If $G \in \mathfrak{N}\mathfrak{A}$ and H is a maximal nilpotent subgroup of G, then the statements* (i)$-$(vi) *of Proposition* 1.8 *are fulfilled.*

Proof. By hypothesis, G' is nilpotent. Therefore by 1.1 and the maximality of H, H is a Carter subgroup of HG', so that H is abnormal in HG'. However, $HG' \trianglelefteq G$, and so $H \leqq H^G \leqq HG'$. Therefore H is abnormal in H^G, so that statement (vi) of 1.8 is fulfilled.

Corollary 1.10. *If $G \in \mathfrak{N}\mathfrak{A}$ and H is any nilpotent subgroup of G, then there is a chain of subgroups $H \leqq J \leqq G$ with H subnormal in J and J abnormal in G.*

That a maximal nilpotent subgroup of a group in $\mathfrak{N}^2$ does not in general satisfy the conditions (i)$-$(vi) of 1.8 we shall see in Example 1 of § 4.

Let $G_1, G_2 \in \mathfrak{N}^2$, and suppose that G_1 has a nilpotent subgroup H_1 which cannot be connected to G_1 by a chain of subgroups $H_1 \leqq J_1 \leqq G_1$, with H_1 subnormal in J_1 and J_1 abnormal in G_1, and that G_2 has a nilpotent subgroup H_2 which cannot be connected to G_2 by a chain of subgroups $H_2 \leqq K_2 \leqq G_2$, with H_2 abnormal in K_2 and K_2 subnormal in G_2. (By 1.10 and 1.4 (i), $G_1 \notin \mathfrak{N}\mathfrak{A}$ and $G_2 \notin \mathfrak{A}\mathfrak{N}$.) Let $G = G_1 \times G_2 \in \mathfrak{N}^2$. Then $H = H_1 \times H_2$ is a nilpotent subgroup of G, and it is easy to see that among the chains (which certainly exist) of subgroups $H \leqq J \leqq K \leqq G$, with H subnormal in J, J abnormal in K and K subnormal in G, there is none in which either $J = H$ or $K = G$.

§ 2. Nilpotent Subgroups of $\mathfrak{N}^3$- and $\mathfrak{N}^4$-groups

In this section we obtain a few extensions of the results of §1 to $\mathfrak{N}^3$- and $\mathfrak{N}^4$-groups.

Proposition 2.1. *If $G \in \mathfrak{N}\mathfrak{A}\mathfrak{N}$ and H is a contranormal nilpotent subgroup of G, then H is contained in a Carter subgroup of G.*

Proof. By hypothesis, there is a normal nilpotent subgroup K of G such that $G/K \in \mathfrak{A}\mathfrak{N}$. Then HK/K is a contranormal nilpotent subgroup of G/K, so that by 1.3, HK/K is a Carter subgroup of G/K. Therefore $HK/K = CK/K$ for some Carter subgroup C of G. Then, since $C \leqq HK$, C is a Carter subgroup of HK. This implies, by conjugacy of Carter subgroups, that every Carter subgroup of HK is also a Carter subgroup of G. The result follows because $H \leqq N_{HK}^{\infty}(H)$, which by 1.1 is a Carter subgroup of HK.

Corollary 2.2. *If $G \in \mathfrak{N}\mathfrak{A}\mathfrak{N}$ and H is any nilpotent subgroup of G, then $a(G:H) \leqq 1$.*

Proof. Let $K = H^{\cdot\cdot G} \in \mathfrak{N}\mathfrak{A}\mathfrak{N}$. Then H is contranormal in K, so that by 2.1, H is contained in a Carter subgroup J of K. Thus $H \leqq J \leqq K \leqq G$, with H subnormal in J, J abnormal in K and K subnormal in G. This shows that $a(G:H) \leqq 1$.

We shall see in Examples 2 and 3 of §4 that a contranormal nilpotent subgroup H of a group G in $\mathfrak{N}^3$ need not be contained in a Carter subgroup of G, and indeed it is possible that $a(G:H) = 2$. However, there is another situation in which we may conclude that $a(G:H) \leqq 1$. To establish this, we make use of the following helpful lemma, a remark made by Professor P. Hall in his lectures in Cambridge in the Lent Term of 1963; his permission to include it here is gratefully acknowledged.

Lemma 2.3. *If H is a subgroup of a group G and $K \trianglelefteqq G$, with H abnormal in HK and HK abnormal in G, then H is abnormal in G.*

Proof. Let $x \in G$. Since HK is abnormal in G,

$$x \in \langle HK, (HK)^x \rangle = K \langle H, H^x \rangle,$$

because $K \trianglelefteqq G$. Let

$$x = ky \quad \text{with} \quad k \in K \quad \text{and} \quad y \in \langle H, H^x \rangle.$$

Since H is abnormal in HK,

$$k \in \langle H, H^k \rangle = \langle H, H^{xy^{-1}} \rangle \leqq \langle H, H^x \rangle.$$

Therefore $x \in \langle H, H^x \rangle$. Hence H is abnormal in G.

Proposition 2.4. *Let $G \in \mathfrak{N}^3$, and suppose that G has abelian system normalizers. If H is a contranormal nilpotent subgroup of G, then $a(G:H) \leqq 1$.*

Proof. There is a normal nilpotent subgroup K of G such that $G/K \in \mathfrak{N}^2$. Then HK/K is a contranormal nilpotent subgroup of G/K, and so by 1.2, HK/K is contained in a Carter subgroup of G/K. Since $G/K \in \mathfrak{N}^2$, the Carter subgroups of G/K coincide with its system normalizers, and therefore (by Hall [5, Theorem 7.3]) $HK/K \leqq DK/K$ for some system normalizer D of G. Then $H \leqq DK$, and by hypothesis, $DK \in \mathfrak{N}\mathfrak{A}$. Therefore, by 1.10, there is a chain of subgroups $H \leqq J \leqq DK$, with H subnormal in J and J abnormal in DK. Since DK/K is abelian it follows that $JK = DK$. Also DK is abnormal in G. Thus $J \leqq JK \leqq G$, with $K \trianglelefteq G$, J abnormal in JK and JK abnormal in G. Hence by 2.3, J is abnormal in G. The result follows.

The hypotheses in 2.4 imply that in fact $G \in \mathfrak{N}^2\mathfrak{A}$. However, it is not enough in 2.4 to suppose merely that $G \in \mathfrak{N}^2\mathfrak{A}$: in Example 2 of § 4, we shall construct a group $G \in \mathfrak{A}\mathfrak{N}\mathfrak{A}$ with a contranormal nilpotent subgroup H such that $a(G:H) = 2$.

2.1 and 2.4 are distinct results. Any non-abelian nilpotent group satisfies the hypotheses of 2.1 but not of 2.4. We shall construct in Example 3 of § 4 a group satisfying 2.4 but not 2.1, a group in which in fact a contranormal nilpotent subgroup is not contained in a Carter subgroup.

Proposition 2.5. *Let $G \in \mathfrak{N}^2\mathfrak{A}\mathfrak{N}$, and suppose that the Carter subgroups of G are abelian. If H is a contranormal nilpotent subgroup of G, then $a(G:H) \leqq 1$.*

Proof. We may assume without loss of generatity that H is a maximal nilpotent subgroup of G. By hypothesis, G has a nilpotent normal subgroup K such that $G/K \in \mathfrak{N}\mathfrak{A}\mathfrak{N}$. Then HK/K is a contranormal nilpotent subgroup of G/K, so that by 2.1, HK/K is contained in a Carter subgroup of G/K. Hence $HK/K \leqq CK/K$ for some Carter subgroup C of G. Now by hypothesis, C is abelian, and so $CK \in \mathfrak{N}\mathfrak{A}$. Therefore since H is a maximal nilpotent subgroup of CK, 1.9 and 1.8 show that $N_{CK}(H)$ is abnormal in CK. It follows, since CK/K is abelian, that $N_{CK}(H)K = CK$. Moreover, CK is abnormal in G. Thus $N_{CK}(H) \leqq N_{CK}(H)K \leqq G$, with $K \trianglelefteq G$, $N_{CK}(H)$ abnormal in $N_{CK}(H)K$ and $N_{CK}(H)K$ abnormal in G. Therefore by 2.3, $N_{CK}(H)$ is abnormal in G. It follows that $a(G:H) \leqq 1$.

Corollary 2.6. *Let G be an A-group of length $\leqq 4$. If H is any abelian subgroup of G, then $a(G:H) \leqq 1$.*

Proof. Let $K = H^{\cdot\cdot G}$. Then K is an A-group of length $\leqq 4$, so that in particular $K \in \mathfrak{A}^4$ and the Carter subgroups of K are abelian. Since H is contranormal in K, it follows from 2.5 that $a(K:H) \leqq 1$. Hence $a(G:H) \leqq 1$.

Remarks. 1. It follows readily from 2.6 and 1.1 that if H is an abelian subgroup of an A-group G of length 5, then $a(G:H) \leqq 2$. In Example 5 of § 4, we shall construct such a G and H with $a(G:H) = 2$. Thus 2.6 cannot be sharpened to yield the same conclusion for A-groups of lengths > 4.

2. Let G be an A-group and H a contranormal abelian subgroup of G. From 2.1, we know that if G is of length $\leqq 3$, then H is contained in a Carter subgroup of G, and from 2.6 that if G has length 4, then $a(G:H)\leqq 1$. In Example 4 of § 4 we shall show, however, that an A-group of length 4 may have a contranormal abelian subgroup not contained in a Carter subgroup.

3. The hypotheses in 2.5 imply that in fact $G\in\mathfrak{N}^2\mathfrak{A}^2$; but it is not enough in 2.5 to suppose merely that $G\in\mathfrak{N}^2\mathfrak{A}^2$, as Example 2 of § 4 shows. Indeed, the hypotheses in 2.5 cannot even be weakened by replacing the supposition that the Carter subgroups of G are abelian by the supposition that the system normalizers of G are abelian. To see this, we may refer to the Example in [6, § 4]; there the group $G\in\mathfrak{A}^4$, the system normalizers of G are abelian and V is a nilpotent subgroup which is subabnormal in G, hence contranormal in G, such that $a(G:V)=2$.

4. Any subgroup containing a system normalizer of a soluble group G is certainly contranormal in G. By a theorem of Carter [4, Theorem 3], if $G\in\mathfrak{N}^3$ and H is any nilpotent subgroup containing a system normalizer of G, then H is contained in a Carter subgroup of G. On the other hand, Examples 2 and 3 of § 4 show that a group in $\mathfrak{N}^3$ may also have contranormal nilpotent subgroups not contained in any Carter subgroup; and the Example in [6, § 4] shows that a group G in $\mathfrak{A}^4$ may have a subabnormal nilpotent subgroup (which certainly contains a system normalizer of G) not contained in a Carter subgroup of G.

5. 2.5 does not contain either of the previous results 2.1, 2.4. Any $\mathfrak{N}\mathfrak{A}\mathfrak{N}$-group with non-abelian Carter subgroups serves to show that 2.1 is not a particular case of 2.5; and the symmetric group of degree 4 is an example of a group satisfying the hypotheses of 2.4 but not of 2.5.

§ 3. Wreath Product Properties

Let X be any group, p any prime number and $W=C_p \wr X$, the wreath product (formed according to the regular representation) of a cyclic group of order p by X. Then

$$W=AX,$$

where

$$A=\mathrm{Dr}\prod_{x\in X}\langle a_x\rangle\trianglelefteq W, \quad A\cap X=1, \quad a_x^p=1 \quad \text{and} \quad a_x^y=a_{xy}$$

for all x and y in X; A is the base group of W. It will be convenient to establish some elementary general properties of W, which we shall subsequently apply in the construction of examples. We retain the notation above throughout this section.

Let Y be any subgroup of X. Any element of A is expressible in the form

$$\prod_{x\in X} a_x^{v_x},$$

where each v_x is an integer such that $0\leqq v_x<p$; and

$$\prod_{x\in X} a_x^{v_x}\in C_A(Y) \quad \text{if and only if} \quad \prod_{x\in X} a_{xy}^{v_x}=\prod_{x\in X} a_x^{v_x}$$

for every element y in Y, that is if and only if $v_x = v_{xy}$ for every x in X and y in Y. Let $\{t_1, \ldots, t_r\}$ be a left transversal to Y in X. Then the preceding observation shows that

$$C_A(Y) = \langle \prod_{y \in Y} a_{t_i y} \mid i = 1, \ldots, r \rangle.$$

Thus we have

3.1.
$$C_A(Y) = \mathrm{Dr} \prod_{i=1}^{r} \langle \prod_{y \in Y} a_{t_i y} \rangle.$$

In particular, $|C_A(Y)| = p^{|X:Y|}$.

Next we consider $C_W(C_A(Y))$. Because A is abelian, surely

$$A \leqq C_W(C_A(Y)) \leqq AX,$$

and so $C_W(C_A(Y)) = A\, C_X(C_A(Y))$. From 3.1, for x in X,

$$x \in C_X(C_A(Y)) \quad \text{if and only if} \quad \prod_{y \in Y} a_{t_i y x} = \prod_{y \in Y} a_{t_i y} \qquad \text{for } i = 1, \ldots, r,$$

and evidently this is true if and only if $x \in Y$.

3.2. $C_X(C_A(Y)) = Y$, *and* $C_W(C_A(Y)) = AY$.

With any subgroup Y of X, we associate now the subgroup

$$Y^* = C_A(Y) \times Y$$

of W. We shall consider the map

$$*\colon\ Y \mapsto Y^*$$

from the set of subgroups of X into the set of subgroups of W, especially when p does not divide $|X|$. Before we impose the latter restriction however, we note that in consequence of 3.2, $*$ is an injective map: for if Y_1, Y_2 are subgroups of X such that $Y_1^* = Y_2^*$, then $A \cap Y_1^* = A \cap Y_2^*$, that is $C_A(Y_1) = C_A(Y_2)$; therefore $C_X(C_A(Y_1)) = C_X(C_A(Y_2))$, and by 3.2, this shows that $Y_1 = Y_2$.

3.3. *The map* $*\colon Y \mapsto Y^*$ *from the set of subgroups of* X *into the set of subgroups of* W *is injective.*

We now assume that p does not divide $|X|$. Then, for any subgroup Y of X, since $|C_A(Y)|$ and $|Y|$ have greatest common divisor 1,

$$N_W(Y^*) = N_W(C_A(Y)) \cap N_W(Y).$$

Consider first $N_W(Y)$. Suppose that $ax \in N_W(Y)$, with a in A and x in X. Then for any y in Y,

$$y^{ax} = y_1 \in Y,$$

so that

$$y^{-1} y^a = y^{-1} y_1^{x^{-1}} \in A \cap X = 1.$$

Therefore $a \in C_A(Y)$ and $x \in N_X(Y)$. Hence

$$N_W(Y) = C_A(Y)\, N_X(Y).$$

We shall show that $N_W(Y) \leqq N_W(C_A(Y))$. To do this, it is enough by the last equation to show that $N_X(Y) \leqq N_W(C_A(Y))$. Suppose then that $x \in N_X(Y)$.

Then for each $i=1, \ldots, r$,

$$\begin{aligned}(\prod_{y\in Y} a_{t_i y})^x &= \prod_{y\in Y} a_{t_i y x} \\ &= \prod_{y\in Y} a_{t_i x y^x} \\ &= \prod_{y\in Y} a_{t_j y' y^x},\end{aligned}$$

where the integer j and element y' of Y are uniquely determined by the equation $t_i x = t_j y'$. Now the map $y \mapsto y' y^x$ is a permutation of Y, and therefore

$$\prod_{y\in Y} a_{t_j y' y^x} = \prod_{y\in Y} a_{t_j y}.$$

Hence by 3.1, $x \in N_W(C_A(Y))$.

The following statement has now been established:

3.4. *If p does not divide $|X|$, then $N_W(Y^*) = N_W(Y) = C_A(Y)\, N_X(Y)$. In particular, $|N_W(Y^*):Y^*| = |N_X(Y):Y|$; and Y^* is self-normalizing in W if and only if Y is self-normalizing in X.*

Next we prove

3.5. *If p does not divide $|X|$, then $(Y^*)^W = A\,Y^X$. In particular, $|W:(Y^*)^W| = |X:Y^X|$; and Y^* is contranormal in W if and only if Y is contranormal in X.*

For this purpose, we note that $A\,Y^X \trianglelefteqq W$ and $(Y^*)^W \geqq Y^X$, so that it is enough to show that $(Y^*)^W \geqq A$. We show first that $(X^*)^W \geqq A$. For each x in X, $x^{-1} x^{a_1} = (a_1^{-1})^x a_1 = a_x^{-1} a_1 \in (X^*)^W$, hence

$$\prod_{x\in X} (a_x^{-1} a_1) = (\prod_{x\in X} a_x)^{-1} a_1^{|X|} \in (X^*)^W.$$

Since

$$\prod_{x\in X} a_x \in X^*,$$

it follows that $a_1^{|X|} \in (X^*)^W$; and therefore, since p does not divide $|X|$, that $a_1 \in (X^*)^W$. This imples that $a_x \in (X^*)^W$ for every x in X, and so $(X^*)^W \geqq A$.

Now we shall show that it follows from this that $(Y^*)^W \geqq A$. Let

$$B_i = \mathrm{Dr} \prod_{y\in Y} \langle a_{t_i y} \rangle, \qquad \text{for } 1 \leqq i \leqq r.$$

Then $A = B_1 \times \cdots \times B_r$. Moreover, for each i, $Y \leqq N_W(B_i)$ and $B_i Y \cong C_p \wr Y$. Therefore, by what has been proved above, with Y replacing X and B_i replacing A, $(C_{B_i}(Y) \times Y)^{B_i Y} \geqq B_i$. But $Y^* \geqq C_{B_i}(Y) \times Y$ for $1 \leqq i \leqq r$, and so $(Y^*)^W \geqq B_1 \times \cdots \times B_r = A$. This establishes 3.5.

If Y_1, Y_2 are conjugate subgroups of X, say $Y_2 = Y_1^x$ with x in X, then $C_W(Y_2) = C_W(Y_1)^x$; and since $A \trianglelefteqq W$, it follows that $C_A(Y_2) = C_A(Y_1)^x$, hence $Y_2^* = (Y_1^*)^x$. Thus the images under the map $*$ of conjugate subgroups of X are conjugate subgroups of W.

Now suppose that Y_1, Y_2 are subgroups of X such that Y_1^*, Y_2^* are conjugate subgroups of W, say $Y_2^* = (Y_1^*)^w$ with w in W. We shall show that Y_1, Y_2

are conjugate subgroups of X (so that Y_1^*, Y_2^* are actually conjugate by an element of X). Since p does not divide $|X|$, the equation $Y_2^* = (Y_1^*)^w$ implies that $C_A(Y_2) = C_A(Y_1)^w$ and $Y_2 = Y_1^w$. Now $w = a\,x$ for some a in A and x in X. Since A is abelian, this gives $C_A(Y_2) = C_A(Y_1)^x$ and hence, since $A \trianglelefteq W$, $C_A(Y_2) = C_A(Y_1^x)$. Then, since $Y_1^x \leqq X$, $Y_1^x \leqq C_X(C_A(Y_2)) = Y_2$, by 3.2. However, $|Y_1^x| = |Y_1^w| = |Y_2|$, and therefore $Y_1^x = Y_2$.

3.6. *If p does not divide $|X|$, then the map $*$: $Y \mapsto Y^*$ is such that subgroups Y_1, Y_2 of X are conjugate in X if and only if Y_1^*, Y_2^* are conjugate in W.*

If Y is a nilpotent subgroup of X, then Y^* is a nilpotent subgroup of W; and on the assumption that p does not divide $|X|$, we shall see that Y^* is in fact a maximal nilpotent subgroup of W. Suppose that $Y^* \leqq U \leqq W$, where U is nilpotent. Then $C_A(Y) \leqq U_p = U \cap A$, and $Y \leqq U^p$. Therefore $U_p \leqq C_A(U^p) \leqq C_A(Y)$, and so $U_p = C_A(Y)$. Hence $U^p \leqq C_W(C_A(Y)) = AY$, by 3.2. Since $Y \leqq U^p$ and p does not divide $|U^p|$, it follows that $U^p = Y$, and therefore that $U = Y^*$.

Next, we shall show that any maximal nilpotent subgroup U of W is conjugate in W to Y^* for some subgroup Y of X. We remark first that, by the Schur-Zassenhaus theorem, U^p is contained in some conjugate in W of X. Hence, replacing U if need be by some conjugate in W, we may suppose that $U^p \leqq X$. Now $U_p = U \cap A \leqq C_A(U^p)$. Therefore $U = U_p \times U^p \leqq C_A(U^p) \times U^p = (U^p)^*$. Since U^p is nilpotent, so also is $(U^p)^*$ nilpotent, and hence it follows from the maximality of U that $U = (U^p)^*$.

In view also of 3.6, we have established

Proposition 3.7. *If p does not divide $|X|$, then the number n of conjugacy classes in W of maximal nilpotent subgroups is equal to the number of conjugacy classes in X of all nilpotent subgroups; and if $Y_1, \dots, Y_n$ are reprensentatives of the distinct conjugacy classes of nilpotent subgroups of X, then $Y_1^*, \dots, Y_n^*$ are representatives of the distinct conjugacy classes of maximal nilpotent subgroups of W.*

We may note that if Y is abelian, then Y^* is also abelian. Thus Y^* is a maximal nilpotent subgroup of W which happens to be abelian; so that in particular it is certainly a maximal abelian subgroup of W. Let U be any maximal abelian subgroup of W. Then as in the argument leading to 3.7, replacing U if necessary by some conjugate in W, we may suppose that $U^p \leqq X$. As before, $U_p = U \cap A \leqq C_A(U^p)$ so that $U = U_p \times U^p \leqq C_A(U^p) \times U^p = (U^p)^*$. Since U^p is abelian so also is $(U^p)^*$ abelian, and hence it follows from the maximality of U that $U = (U^p)^*$. This yields

Proposition 3.8. *If p does not divide $|X|$, then every maximal abelian subgroup of W is also a maximal nilpotent subgroup of W. The number m of conjugacy classes in W of maximal abelian subgroups is equal to the number of conjugacy classes in X of all abelian subgroups, and if $Y_1, \dots, Y_m$ are representatives of the distinct conjugacy classes of abelian subgroups of X, then $Y_1^*, \dots, Y_m^*$ are representatives of the distinct conjugacy classes of maximal abelian subgroups of W.*

§ 4. Construction of Examples

As well as the results of § 3, we shall need two simple lemmas, which will be established before any construction of examples is undertaken. It is a familiar fact that if H is a maximal nilpotent subgroup of a group G, then $N_G(H)$ is self-normalizing in G; but a slightly stronger assertion than this can be made, namely that any subgroup of G in which H is subnormal is contained in $N_G(H)$, or in the terminology of Carter [4], that $N_G(H)$ is the *subnormalizer* of H in G.

Lemma 4.1. *Let H be a maximal nilpotent subgroup of a group G. Then $N_G(H)$ is the subnormalizer of H in G.*

Proof. Suppose that H is subnormal in a subgroup J of G. Then H lies in the Fitting subgroup of J. Since H is a maximal nilpotent subgroup of J, it follows that H coincides with the Fitting subgroup of J. Therefore $J \leqq N_G(H)$.

The second lemma concerns the structure of a system normalizer of a soluble group which has a normal Sylow subgroup.

Lemma 4.2. *Suppose that G is a soluble group with a normal Sylow p-subgroup P. Let G_1 be a p-complement of G and let D_1 be a system normalizer of G_1. Then $C_P(G_1) \times D_1$ is a system normalizer of G.*

Proof. Consider a Sylow system of G which includes the p-complement G_1. If D is the corresponding system normalizer of G, then

$$D_p = P \cap N_G(G_1) = C_P(G_1) \qquad \text{(Hall [5, Theorem 3.3])}.$$

Furthermore $D^p \leqq G_1$, and so ([7, Lemma 5.1]) D^p is a system normalizer of G_1. Therefore $D^p = D_1^x$ for some x in G_1. Then

$$D = C_P(G_1) \times D_1^x = \left(C_P(G_1) \times D_1\right)^x,$$

since x centralizes $C_P(G_1)$. Hence $C_P(G_1) \times D_1$ $(= D^{x^{-1}})$ is a system normalizer of G.

Examples. 1. We shall construct a group G in $\mathfrak{A}\mathfrak{N}$ with a maximal nilpotent subgroup H not satisfying the equivalent conditions (i)–(vi) of 1.8. Let $G = C_p \wr D_8$, where p is an odd prime and D_8 is a dihedral group of order 8. Then $G = AX$, where A is the base group of the wreath product, and $X = \langle x, y \rangle$ with $x^4 = y^2 = (xy)^2 = 1$. Certainly $G \in \mathfrak{A}\mathfrak{N}$. Let $Y = \langle y \rangle$ and let $H = Y^* = C_A(Y) \times Y$. By 3.7, H is a maximal nilpotent subgroup of G. By 3.4, $N_G(H) = C_A(Y)\, N_X(Y)$: this is not contranormal in G, because $N_G(H) \leqq A\, N_X(Y) \lhd G$, since $N_X(Y) \lhd X$. Therefore H does not satisfy the conditions of 1.8.

2. Next, we shall construct a group G in $\mathfrak{A}\mathfrak{N}\mathfrak{A}$ with a contranormal nilpotent subgroup H such that $a(G:H) = 2$. With this goal in view, we begin by letting X be the subgroup of the general linear group $GL_{2n}(p)$ discussed in [6, § 3] (denoted there by G), where n is a positive integer and p is an odd prime:

$$X = \langle U, t \rangle,$$

where

$$U = \{1 + \sum_{i<j} \lambda_{ij} e_{ij} | \lambda_{ij} \in GF(p)\},$$

the group of $2n$-square unitriangular matrices with entries in $GF(p)$ and

$$t = \operatorname{diag}\{-1, +1, -1, +1, \ldots, -1, +1\} = \sum_{i=1}^{2n} (-1)^i e_{ii}.$$

(The notation is as in [6].) Then

$$e_{ij}^t = (-1)^{i+j} e_{ij},$$

so that

$$(1 + \sum_{i<j} \lambda_{ij} e_{ij})^t = 1 + \sum_{i<j} (-1)^{i+j} \lambda_{ij} e_{ij};$$

and this is equal to

$$1 + \sum_{i<j} \lambda_{ij} e_{ij} \qquad \text{if and only if } (-1)^{i+j} \lambda_{ij} = \lambda_{ij} \text{ for all } i \text{ and } j,$$

that is if and only if $\lambda_{ij} = 0$ whenever $i+j$ is odd. Thus

$$C_U(t) = \{1 + \sum_{i<j} \lambda_{ij} e_{ij} | \lambda_{ij} = 0 \text{ whenever } i+j \text{ is odd}\}.$$

Now by Lemma 4.2, $C_U(t) \times \langle t \rangle$ is a system normalizer D of X. Moreover it was shown in [6] that $\langle t \rangle^X = X$. Thus any subgroup of X lying between $\langle t \rangle$ and D is a contranormal nilpotent subgroup of X.

Now we prescribe that $n \geqq 3$. This ensures that $C_U(t)$ is non-abelian, for $1 + e_{13}$ and $1 + e_{35}$ both belong to $C_U(t)$, and

$$(1 + e_{13})(1 + e_{35}) = 1 + e_{13} + e_{15} + e_{35},$$

whereas

$$(1 + e_{35})(1 + e_{13}) = 1 + e_{13} + e_{35}.$$

It follows, since $C_U(t)$ is a p-group and p is an odd prime, that there is a subgroup V of $C_U(t)$ which is not normal in $C_U(t)$. Let $Y = V \times \langle t \rangle$, a contranormal nilpotent subgroup of X. Since

$$X = U \langle t \rangle,$$

$$N_X(Y) = N_U(Y) \langle t \rangle;$$

and $N_U(Y) = N_U(V) \cap C_U(t) = N_{C_U(t)}(V) < C_U(t)$, by choice of V. Therefore $N_X(Y) < D$, and so, since D is nilpotent, $N_X(Y)$ is a proper subgroup of its normalizer in X.

Let q be an odd prime different from p, and let $G = C_q \wr X$. Then $G \in \mathfrak{A}\mathfrak{N}\mathfrak{A}$. Let A denote the base group of G, and let

$$H = Y^* = C_A(Y) \times Y.$$

By 3.7, H is a maximal nilpotent subgroup of G; and by 3.5 H is contranormal in G. Therefore, by Lemma 4.1, we could have $a(G:H) \leqq 1$ only if $N_G(H)$ were abnormal in G. But by 3.4, $N_G(H) = C_A(Y) N_X(Y) \leqq A N_X(Y)$; and since $A \trianglelefteq G$,

$A \cap X = 1$ and $N_X(Y) < N_X(N_X(Y))$, $A N_X(Y) < N_G(A N_X(Y))$. Thus $N_G(H)$ is not abnormal in G, and so, since $a(G:H) \leqq 2$ ([6, Theorem 1]) it follows that $a(G:H) = 2$.

3. In Example 2, the system normalizers of G are non-abelian: we know by Proposition 2.4 that if a group G in $\mathfrak{N}^3$ has abelian system normalizers, then $a(G:H) \leqq 1$ for any contranormal nilpotent subgroup H of G. We shall show, however, that H need not be contained in a Carter subgroup of G.

For this purpose, let X be the split extension of a quaternion group by a cyclic group of order 3 described in Remark 1 following 1.6. Let p be a prime > 3, and let $G = C_p \wr X$. Then $G \in \mathfrak{N}^3$. Let Y be a subgroup of X of order 3, and let $V = N_X(Y)$: then V is cyclic of order 6, and is both a Carter subgroup and a system normalizer of X. Let A be the base group of G. Then by Lemma 4.2, $C_A(X) \times V$ is a system normalizer of G, and so the system normalizers of G are abelian.

By 3.4, $V^* = C_A(V) \times V$ is a Carter subgroup of G. Also Y is contranormal in X, and so by 3.5, $Y^* = C_A(Y) \times Y$ is a contranormal nilpotent subgroup of G. Since $|V^*| \neq |Y^*|$, Y^* is not a Carter subgroup of G, and since by 3.7, Y^* is a maximal nilpotent subgroup of G, Y^* is not contained in a Carter subgroup of G.

Thus G is a group satisfying the hypotheses of 2.4 and having a contranormal nilpotent subgroup not contained in a Carter subgroup of G. Since V^* is abelian, G also satisfies the hypotheses of 2.5, and therefore shows that in 2.5 H need not be contained in a Carter subgroup of G. We see this also from the following example.

4. We shall show how to construct an A-group of length 4 with a contranormal abelian subgroup not contained in a Carter subgroup. We begin by selecting any A-group X of length 3 with a system normalizer D and Carter subgroup E such that $D < E$. (For instance, we may take X to be as in Example 5 below.) Then let p be a prime not dividing $|X|$, and consider $W = C_p \wr X$. It is clear that W is an A-group of length 4. Let P denote the base group of W. By 3.7, $D^* = C_P(D) \times D$ is a maximal nilpotent subgroup of W which, by 3.5, is contranormal in W, because D is contranormal in X. Also, by 3.4, $E^* = C_P(E) \times E$ is a Carter subgroup of W. Since $D < E$, D^* and E^* belong to different conjugacy classes in W, and therefore D^* is not a Carter subgroup of W. Thus D^* is a contranormal abelian subgroup of W which is not contained in a Carter subgroup of W.

5. Finally, we shall show that an A-group of length 5 can have an abelian subgroup H with $a(G:H) = 2$. (We know as a consequence of Corollary 2.6 that $a(G:H) \leqq 2$.) We observe first that it will be enough for this purpose to construct an A-group W of length 4 with an abelian subgroup V such that

(i) V is contranormal in W, and

(ii) $N_W(V)$ is not abnormal in W.

For then we can choose a prime q which does not divide $|W|$, and let $G = C_q \wr W$ and $H = V^* = C_Q(V) \times V$, where Q is the base group of G. Then G is an A-group of length 5. By 3.7, H is a maximal nilpotent subgroup of G, and by 3.5, (i) implies that H is contranormal in G. Then, as in Example 2, it follows from Lemma 4.1 that $a(G:H) \leqq 1$ only if $N_G(H)$ is abnormal in G. By 3.4, $N_G(H) = C_Q(V)\, N_W(V) \leqq Q\, N_W(V)$, and (ii) implies that $Q\, N_W(V)$ is not abnormal in G. Hence $a(G:H) > 1$ and so $a(G:H) = 2$.

We shall now construct such a group W. Let X be a split extension of an elementary abelian group $B = \langle b \rangle \times \langle c \rangle$, of order 5^2, by a symmetric group $S = \langle u, v \rangle$ of degree 3, defined by the relations $b^5 = c^5 = u^3 = v^2 = (u\,v)^2 = 1$, $b\,c = c\,b$, $b^u = c$, $c^u = b^{-1} c^{-1}$, $b^v = c$, $c^v = b$. Then X is an A-group of length 3. By Lemma 4.2, $C_B(S) \times \langle v \rangle$ is a system normalizer D of X. Now $C_B(S) \leqq Z(X) = 1$, so that in fact $D = \langle v \rangle$. We see that $C_B(v) = \langle b\,c \rangle$; let $E = \langle b\,c \rangle \times \langle v \rangle$. (Actually, E is the (by Carter [4, Theorem 2]) unique Carter subgroup of X containing D.)

Let p be a prime > 5, and let $W = C_p \wr X$: then W is an A-group of length 4. Let P denote the base group of W:

$$P = \mathrm{Dr} \prod_{x \in X} \langle a_x \rangle,$$

where $a_x^p = 1$ for every x in X. By Lemma 4.2, $C_P(X) \times D$ is a system normalizer $\tilde{D}$ of W, and by 3.1, $C_P(X) = \langle z \rangle$, where

$$z = \prod_{x \in X} a_x.$$

Let $P_1 = \langle z \rangle \times \langle a_1\, a_v \rangle \leqq C_P(D)$, because $(a_1\, a_v)^v = a_v\, a_1$; and let $V = P_1 \times D$. Then V is abelian, and is contranormal in W because it contains a system normalizer $\tilde{D}$ of W.

It remains to show that condition (ii) is satisfied. Since p does not divide $|X|$,

$$N_W(V) = N_W(P_1) \cap N_W(D).$$

Certainly $P \leqq N_W(P_1) \leqq P\,X$, so that

$$N_W(P_1) = P\, N_X(P_1);$$

and by 3.4,

$$N_W(D) = C_P(D)\, N_X(D).$$

For any x in X, $(a_1\, a_v)^x = a_x\, a_{v\,x}$, and this belongs to P_1 if and only if $a_x\, a_{v\,x} = a_1\, a_v$, hence if and only if $x \in \langle v \rangle = D$. Thus

$$N_X(P_1) = D,$$

and so

$$N_W(V) = (P\,D) \cap \big(C_P(D)\, N_X(D)\big) = C_P(D)\, D = D^*,$$

the maximal nilpotent subgroup of W corresponding by 3.7 to the nilpotent subgroup D of X. Now D is not self-normalizing in X, for $D < E$, and so by 3.4, D^* is not self-normalizing in W. This shows that $N_W(V)$ is not abnormal in W. Hence condition (ii) is satisfied and the construction is complete.

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Dr. John S. Rose
Department of Pure Mathematics
The University of Newcastle
Newcastle upon Tyne, Great Britain

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Absolutely faithful group actions

By JOHN S. ROSE

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Absolutely faithful group actions

BY JOHN S. ROSE
The University, Newcastle upon Tyne

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1. *The definition and main result.* It has been shown ((1), § 3) that if G is any finite group and p any prime number not dividing $|G|$, then the number of conjugacy classes of maximal nilpotent subgroups in the regular wreath product of a cyclic group of order p by G is equal to the number of conjugacy classes of all nilpotent subgroups in G. This fact, together with various properties of the map by means of which it was established, proved helpful in dealing with questions of construction raised in (1). The present note isolates the key property of the wreath product on which the argument rests, and from this shows how the argument can be carried over to a more general context. The essential situation is that a group G acts on a group A in a way which will be called 'absolutely faithful'.

A group G is said to *act* on a group A if for each element g of G and each element a of A, there is a uniquely determined element a^g of A, such that

$$\text{(i)}\ (a_1a_2)^g = a_1^g a_2^g, \quad \text{(ii)}\ a^{g_1g_2} = (a^{g_1})^{g_2}, \quad \text{(iii)}\ a^1 = a$$

for all a, a_1, $a_2 \in A$ and g, g_1, $g_2 \in G$. (Such an action naturally determines a homomorphism of G into the automorphism group $\operatorname{Aut} A$ of A, and conversely any homomorphism of G into $\operatorname{Aut} A$ determines an action of G on A.) For each subgroup H of G, set

$$C_A(H) = \{a \in A \mid a^h = a \quad \text{for all} \quad h \in H\},$$

that is the 'fixed point subgroup of A under H' or the 'centralizer of H in A'. Similarly, for each subgroup B of A, set

$$C_G(B) = \{g \in G \mid b^g = b \quad \text{for all} \quad b \in B\},$$

the 'centralizer of B in G'. The action of G on A determines a semi-direct product K of A by G; in K, $a^g = g^{-1}ag$ and $C_A(H)$, $C_G(B)$ have their customary meanings as centralizers.

Definition. Suppose that the group G acts on the group A. The action is called *absolutely faithful* if, for every subgroup H of G, $C_G(C_A(H)) = H$.

When this condition is satisfied, then in particular $C_G(A) = C_G(C_A(1)) = 1$, so that the action of G on A is faithful. Of course, faithful actions are not in general absolutely faithful. We begin by noting an equivalent condition for an action to be absolutely faithful.

1·1. *Suppose that the group G acts on the group A. The action is absolutely faithful if and only if the map $H \mapsto C_A(H)$ from the set of subgroups of G into the set of subgroups of A is injective.*

Proof. If the action of G on A is absolutely faithful and if H_1, H_2 are subgroups of G such that $C_A(H_1) = C_A(H_2)$, then $H_1 = C_G(C_A(H_1)) = C_G(C_A(H_2)) = H_2$. Conversely, suppose that the map $H \mapsto C_A(H)$ is injective. Let $\bar{H} = C_G(C_A(H))$. Then $H \leqslant \bar{H}$, from which it follows that $C_A(H) \geqslant C_A(\bar{H})$. But also, by definition, $\bar{H}$ centralizes $C_A(H)$ and therefore $C_A(H)$ centralizes $\bar{H}$, that is $C_A(H) \leqslant C_A(\bar{H})$. Hence $C_A(H) = C_A(\bar{H})$, and so by hypothesis $H = \bar{H}$. Therefore the action is absolutely faithful.

For the remainder of this section, we suppose that the group G acts absolutely faithfully on the group A, and we denote by K the semi-direct product of A by G determined by this action. Further conditions are imposed as they are required. The object of discussion is the map

$$*: H \mapsto H^* = C_A(H) \times H$$

from the set of subgroups of G into the set of subgroups of K.

1·2. *The map $*$ is injective.*

Proof. Suppose that H_1, H_2 are subgroups of G such that $H_1^* = H_2^*$. Then

$$C_A(H_1) = A \cap H_1^* = A \cap H_2^* = C_A(H_2),$$

and so by 1·1, $H_1 = H_2$.

1·3. *If A is abelian, then for every subgroup H of G, $C_K(C_A(H)) = AH$.*

Proof. Since A is abelian, $A \leqslant C_K(C_A(H)) \leqslant K = AG$. The assertion follows by Dedekind's rule and the hypothesis that the action of G on A is absolutely faithful.

1·4. *Suppose that A is abelian, and let H_1, H_2 be subgroups of G. Then H_1^*, H_2^* are conjugate in K if and only if H_1, H_2 are conjugate in G.*

Proof. Suppose first that $H_1^g = H_2$, for some $g \in G$. Then $C_K(H_1)^g = C_K(H_2)$, and therefore, since A is a normal subgroup of K, $C_A(H_1)^g = C_A(H_2)$. Hence $(H_1^*)^g = H_2^*$. Conversely, suppose that $(H_1^*)^k = H_2^*$, for some $k \in K$. Then, since A is normal in K, $(A \cap H_1^*)^k = A \cap H_2^*$, that is $C_A(H_1)^k = C_A(H_2)$. Now $k = ag$ with suitable elements a in A and g in G. Since A is abelian and normal in K, it follows that $C_A(H_1^g) = C_A(H_2)$. Then by 1·1, $H_1^g = H_2$.

In order to carry over the arguments in (1), we now restrict attention to finite groups acting in a 'relatively prime' way. Then we can formulate the main result as

Proposition 1·5. *Let K be a finite group with an abelian normal Hall subgroup A, and let G be a complement of A in K. Suppose that the action (by conjugation) of G on A is absolutely faithful. If $H_1, \ldots, H_n$ are representatives of all the distinct conjugacy classes of nilpotent subgroups of G, then $H_1^*, \ldots, H_n^*$ are representatives of all the distinct conjugacy classes of maximal nilpotent subgroups of K; where, for any subgroup H of G,*

$$H^* = C_A(H) \times H.$$

Proof. Let π denote the set of prime divisors of $|A|$. For any nilpotent subgroup L of K, let L_π denote the unique Hall π-subgroup of L and $L_{\pi'}$ the unique Hall

π'-subgroup of L, so that $L = L_\pi \times L_{\pi'}$. For any nilpotent subgroup H of G, H^* is certainly nilpotent. Suppose that L is a nilpotent subgroup of K with $H^* \leqslant L$. Then

$$C_A(H) \leqslant L_\pi = A \cap L \quad \text{and} \quad H \leqslant L_{\pi'}.$$

Hence $$L_\pi \leqslant C_A(L_{\pi'}) \leqslant C_A(H), \quad \text{and so} \quad L_\pi = C_A(H).$$

Therefore $$L_{\pi'} \leqslant C_K(C_A(H)) = AH, \quad \text{by } 1{\cdot}3.$$

Since $H \leqslant L_{\pi'}$ and $(|A|, |L_{\pi'}|) = 1$, it follows that $L_{\pi'} = H$. Therefore

$$H^* = C_A(H) \times H = L_\pi \times L_{\pi'} = L.$$

Thus H^* is a maximal nilpotent subgroup of K.

In order to complete the proof, it will be enough, in view of 1·4, to show that every maximal nilpotent subgroup of K is conjugate in K to some H^*, with H a nilpotent subgroup of G. Suppose then that L is a maximal nilpotent subgroup of K. By the Schur–Zassenhaus theorem $L_{\pi'}$ lies in some conjugate in K of G. Therefore we may assume that $L_{\pi'} \leqslant G$. Now $L_\pi = A \cap L \leqslant C_A(L_{\pi'})$, and so

$$L = L_\pi \times L_{\pi'} \leqslant C_A(L_{\pi'}) \times L_{\pi'} = L^*_{\pi'}.$$

But $L_{\pi'}$ is nilpotent, therefore $L^*_{\pi'}$ is nilpotent, and hence by the maximality of L, $L = L^*_{\pi'}$.

Corollary 1·6. *Let K, A, G be as in* 1·5, *with the same hypotheses. Let $\mathfrak{X}$ be a class of finite nilpotent groups containing the class of finite abelian groups and such that* (i) *any Hall subgroup of an $\mathfrak{X}$-group is an $\mathfrak{X}$-group and* (ii) *the direct product of any two $\mathfrak{X}$-groups of coprime orders is an $\mathfrak{X}$-group. If $H_1, \ldots, H_m$ are representatives of all the distinct conjugacy classes of $\mathfrak{X}$-subgroups of G, then $H_1^*, \ldots, H_m^*$ are representatives of all the distinct conjugacy classes of maximal $\mathfrak{X}$-subgroups of K. In particular, every maximal $\mathfrak{X}$-subgroup of K is also a maximal nilpotent subgroup of K.*

Proof. If H is any $\mathfrak{X}$-subgroup of G, then, since $C_A(H)$ is abelian and

$$(|C_A(H)|, |H|) = 1,$$

H^* is an $\mathfrak{X}$-subgroup of K. Moreover, since it is a maximal nilpotent subgroup of K, H^* must certainly be a maximal $\mathfrak{X}$-subgroup of K. By 1·4, the proof will be completed by showing that every maximal $\mathfrak{X}$-subgroup of K is conjugate in K to some H^*, with H an $\mathfrak{X}$-subgroup of G. Now a maximal $\mathfrak{X}$-subgroup L of K must lie in some maximal nilpotent subgroup of K and therefore, by 1·5 and replacing L if need be by some conjugate in K, we may suppose that $L \leqslant H^*$, where H is some nilpotent subgroup of G. With the previous notation, it follows that

$$L_\pi \leqslant C_A(H) \quad \text{and} \quad L_{\pi'} \leqslant H.$$

Then $C_A(H) \leqslant C_A(L_{\pi'})$ and so $L \leqslant L^*_{\pi'}$. However, $L_{\pi'}$ is a Hall subgroup of L and consequently an $\mathfrak{X}$-subgroup of G, and therefore $L^*_{\pi'}$ is an $\mathfrak{X}$-subgroup of K. Then the maximality of L implies that $L = L^*_{\pi'}$.

The condition for the action in 1·5 to be absolutely faithful can be simply expressed by 1·1 if G is cyclic of prime power order. Then we have

Corollary 1·7. *Let p be a prime number. Suppose that the finite group K has an abelian normal p-complement A and that the Sylow p-subgroups of K are cyclic of order p^n. Let $\langle g\rangle$ be a Sylow p-subgroup of K. If*

$$C_A(g) < C_A(g^p) < \dots < C_A(g^{p^{n-1}}) < C_A(g^{p^n}) = A,$$

then K has precisely $n+1$ conjugacy classes of maximal abelian subgroups, of which representatives are the subgroups $C_A(g^{p^j}) \times \langle g^{p^j}\rangle$, with $j = 0, 1, \dots, n$.

2. *Permutation representations.* Suppose that a group G *permutes* a set X: that is, for each element g of G and each element x of X, there is a uniquely determined element xg of X, such that (i) $x(g_1g_2) = (xg_1)g_2$ and (ii) $x1 = x$ for all $x \in X$ and $g_1, g_2 \in G$. (These conditions determine naturally a homomorphism of G into the unrestricted symmetric group S_X of X, and conversely G permutes X by means of any homomorphism of G into S_X.) For any group B, there is a corresponding action of G on the (restricted) direct power $A = \mathrm{Dr}\, B^X$, defined by

$$f^g(x) = f(xg^{-1}) \quad \text{for all} \quad f \in A, \quad g \in G \quad \text{and} \quad x \in X.$$

The semi-direct product of A by G determined by this action is of course the wreath product of B by G defined by the given permutation representation of G. We ask: when is the action of G on A absolutely faithful?

Let H be any subgroup of G, and let $f \in A$. Then $f \in C_A(H)$ if and only if $f(xh^{-1}) = f(x)$ for all $h \in H$ and $x \in X$. As G permutes X, so also H permutes X and therefore X is partitioned into H-orbits. The condition for f to belong to $C_A(H)$ is precisely that f assume constant values on all H-orbits. Let the distinct H-orbits be denoted by X_i $(i \in I)$, with $\bigcup_{i \in I} X_i = X$. Then we have

$$C_A(H) = \mathrm{Dr}_{i \in I}\, B_i,$$

where $\quad B_i = \{f \in A \mid f(x) = f(y) \quad \text{for all} \quad x, y \in X_i \quad \text{and} \quad f(x) = 1 \quad \text{for all} \quad x \notin X_i\}.$

Each B_i is isomorphic to B.

Suppose now that $|B| \neq 1$. Let $g \in G$. We show that $g \in C_G(C_A(H))$ if and only if g fixes every H-orbit of X, that is $X_i g = X_i$ for every $i \in I$. Suppose first that g fixes every H-orbit. Then if $f \in C_A(H)$ and if $x \in X_i$,

$$f^g(x) = f(xg^{-1}) = f(x),$$

because $xg^{-1} \in X_i$ and f assumes a constant value on X_i. This is true for all $i \in I$, and so $f^g = f$. Hence $g \in C_G(C_A(H))$. Now suppose that g does not fix every H-orbit: then there is a $j \in I$ and an element $y \in X_j$ such that $yg \notin X_j$. We choose any non-identity element b of B, and define $f \in B_j$ by

$$f(x) = \begin{cases} b & \text{for all} \quad x \in X_j, \\ 1 & \text{for all} \quad x \in X - X_j. \end{cases}$$

Then $\qquad f^g(yg) = f(y) = b, \quad \text{since} \quad y \in X_j,$

but $\qquad f(yg) = 1, \qquad \text{since} \quad yg \notin X_j.$

Therefore $f^g \neq f$, and so $g \notin C_G(C_A(H))$.

Now if g fixes every H-orbit of X, then for each element x of X, because xg belongs to the same H-orbit as x there is an element h of H such that $xg = xh$. Thus $g \in \mathrm{Stab}_G(x)\,H$, where $\mathrm{Stab}_G(x)$ denotes the stabilizer in G of x. Conversely, if $g \in \mathrm{Stab}_G(x)\,H$ for every element x of X, then g fixes every H-orbit of X.

The following result has thus been established.

PROPOSITION 2·1. *Suppose that the group G permutes the set X, and let B be any group with $|B| \neq 1$. Under the corresponding action of G on the group $A = \mathrm{Dr}\,B^X$, for every subgroup H of G*

$$C_G(C_A(H)) = \bigcap_{x\in X} (\mathrm{Stab}_G(x)\,H).$$

Thus the action is absolutely faithful if and only if $\bigcap_{x\in X} (\mathrm{Stab}_G(x)\,H) = H$, for every subgroup H of G.

When $X = G$ and G permutes itself according to the regular representation, then $\mathrm{Stab}_G(x) = 1$ for every element x. Thus we can deduce from 2·1 and 1·5 the following generalization of Proposition 3·7 of (1).

COROLLARY 2·2. *Let π be a set of prime numbers, and let B be a finite abelian π-group, with $|B| \neq 1$, and G a finite π'-group. Let K be the regular wreath product of B by G, and let $A = \mathrm{Dr}\,B^G$, the base group of K. For each subgroup H of G, define the subgroup H^* of K by $H^* = C_A(H) \times H$. If $H_1, \ldots, H_n$ are representatives of all the distinct conjugacy classes of nilpotent subgroups of G, then $H_1^*, \ldots, H_n^*$ are representatives of all the distinct conjugacy classes of maximal nilpotent subgroups of K.*

If the action in 2·1 is absolutely faithful, then in particular $\bigcap_{x\in X} \mathrm{Stab}_G(x) = 1$, that is the associated permutation representation of G on X is faithful. On the other hand, faithful permutation representations do not in general lead to absolutely faithful group actions, as the following example shows.

EXAMPLE 2·3. In 2·1, let $X = \{1, 2, 3\}$ and let $G = S_3$, the symmetric group of degree 3: the natural permutation representation of G on X is faithful. Then

$$\mathrm{Stab}_G(1) = \langle (23) \rangle = G_1, \quad \text{say;} \quad \mathrm{Stab}_G(2) = \langle (31) \rangle = G_2;$$

$\mathrm{Stab}_G(3) = \langle (12) \rangle = G_3$. Let $H = \langle (123) \rangle < G$. Then $G_i H = G$ for $i = 1, 2, 3$. By 2·1,

$$C_G(C_A(H)) = \bigcap_{i=1}^{3} G_i H = G \neq H.$$

Therefore for any non-trivial group B, the natural action of S_3 on the direct product $B_1 \times B_2 \times B_3$ of 3 copies of B is not absolutely faithful.

However, transitive permutative representations other than regular representations can lead to absolutely faithful group actions.

EXAMPLE 2·4. In 2·1, let $G = A_4$, the alternating group on the set $\{1, 2, 3, 4\}$, and let X be the set of right cosets in G of the subgroup $K = \langle (12)(34) \rangle$ of order 2. G permutes the elements of X transitively by right multiplication, and the associated permutation representation (of degree 6) is faithful. For each $x \in G$, $\mathrm{Stab}_G(Kx) = x^{-1}Kx$, so that by 2·1 the condition that the corresponding action of G on $\mathrm{Dr}\,B^X$ be absolutely faithful, for any non-trivial group B, is this: $\bigcap_{x\in G} x^{-1}KxH = H$ for every subgroup H of G. The condition is clearly satisfied when H is any subgroup of G containing some conjugate in G of K; and it is also satisfied when $H = 1$. It only remains to consider subgroups H of order 3: then each $x^{-1}KxH$ is a subset of G with $|x^{-1}KxH| = 6$; since $\bigcap_{x\in G} x^{-1}KxH$ is a subgroup of G containing H, and since G has no subgroup of order 6, it follows that the condition $\bigcap_{x\in G} x^{-1}KxH = H$ is satisfied in this case too. Therefore the corresponding action is absolutely faithful.

3. *Normalizers and normal closures.* For the applications in (1), it was a convenience that the effect of the map $*$ on normalizers and normal closures could be very simply described. This is also true in a more general situation; indeed for this generalization the hypothesis of an absolutely faithful group action is not needed.

We suppose that the group G acts on the group A, and denote by K the semi-direct product of A by G determined by this action. For each subgroup H of G, we define as before

$$H^* = C_A(H) \times H.$$

We are interested in the normalizer $N_K(H^*)$ and the normal closure $(H^*)^K$ of H^* in K. We note first

3·1. *For every subgroup H of G, $N_K(H) = C_A(H)\,N_G(H)$.*

Proof. Let $ag \in N_K(H)$, with $a \in A$ and $g \in G$. Then for each $h \in H$,

$$h^{ag} = h_1 \in H,$$

and so

$$h^{-1}h^a = h^{-1}h_1^{g^{-1}} \in A \cap G = 1.$$

Therefore $a \in C_A(H)$ and $g \in N_G(H)$.

3·2. *For every subgroup H of G, $N_G(H) \leqslant N_G(C_A(H))$.*

Proof. Let $g \in N_G(H)$ and $a \in C_A(H)$. Then for each h in H,

$$(a^g)^h = (a^{ghg-1})^g = a^g, \quad \text{since} \quad ghg^{-1} \in H.$$

Therefore $a^g \in C_A(H)$.

Now if A and G are finite and $(|A|, |G|) = 1$, then

$$N_K(H^*) = N_K(C_A(H)) \cap N_K(H) = N_K(H),$$

by 3·1 and 3·2. Thus 3·1 yields the following generalizations of 3·4 of (1):

PROPOSITION 3·3. *Let K be a finite group with a normal Hall subgroup A, and let G be a complement of A in K. Then for each subgroup H of G,*

$$N_K(H^*) = N_K(H) = C_A(H)\,N_G(H).$$

In particular,

$$|N_K(H^*) \colon H^*| = |N_G(H) \colon H|.$$

For the corresponding generalization of 3·5 of (1), we assume that A is abelian. We prove

PROPOSITION 3·4. *Let K be a finite group with an abelian normal Hall subgroup A, and let G be a complement of A in K. Then, for each subgroup H of G, $(H^*)^K = AH^G$. In particular, $|K \colon (H^*)^K| = |G \colon H^G|$.*

Proof. Certainly AH^G is a normal subgroup of K, and $H^G \leqslant (H^*)^K$; hence it is enough to show that $A \leqslant (H^*)^K$. Since A is abelian, $\prod_{h\in H} a^h$ is, for any $a \in A$, an unambiguously defined element of A. We observe that it belongs to $C_A(H)$: for if h' is any element of H, then

$$(\prod_{h\in H} a^h)^{h'} = \prod_{h\in H} a^{hh'} = \prod_{h\in H} a^h,$$

since $h \mapsto hh'$ is a permutation of H and A is abelian. Now for any $a \in A$ and $h \in H$,

$$(a^h)^{-1} a = h^{-1}(a^{-1}ha) \in (H^*)^K.$$

Hence

$$\prod_{h\in H} ((a^h)^{-1} a) \in (H^*)^K;$$

that is, since A is abelian,

$$(\prod_{h\in H} a^h)^{-1}\, a^{|H|} \in (H^*)^K.$$

But $\prod_{h\in H} a^h \in H^*$, and therefore

$$a^{|H|} \in (H^*)^K.$$

Since $(|A|, |H|) = 1$, it follows that

$$a \in (H^*)^K.$$

Thus $A \leqslant (H^*)^K$, as we wished to show.

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On the Splitting of Extensions by a Group of Prime Order

JOHN S. ROSE

To Helmut Wielandt, on his sixtieth birthday, 19 December, 1970

In this note, p always stands for a prime number. Necessary and sufficient conditions on a group K are given for all extensions of K by a group of order p to split. As an illustration of the general result, the case in which K is a non-abelian p-group with a cyclic subgroup of index p is considered.

The starting point for this enquiry was a lemma proved in [2] (Lemma 2.1):

(I) *Let K and X be groups and Y a central subgroup of X. Suppose that all extensions of K by X/Y split. Then any isomorphism of Y onto a central subgroup of K can be extended to a homomorphism of X into K.*

The following fact is a direct consequence ([2], Corollary 2.2):

(II) *Let K be a finite group and let n be the largest integer such that K contains an element of order p^n. If $Z(K)$ has order divisible by p then there is a non-split extension of K by a cyclic group of order p^n.*

(Here and elsewhere $Z(K)$ denotes the centre of K.)
Another simple consequence of (I) is

(III) *If K is a group which has an element in $Z(K)$ which is not the p-th power of any element in K, then there is a non-split extension of K by a group of order p.*

To make this deduction, suppose that z is any element of $Z(K)$. Let X be a cyclic group of order p times the order of z (infinite if z has infinite order), say $X=\langle x\rangle$. Then there is an isomorphism φ of the (central) subgroup $\langle x^p\rangle$ of X onto the central subgroup $\langle z\rangle$ of K, with $x^p\varphi=z$. If all extensions of K by a group of order p split, then from (I), since $X/\langle x^p\rangle$ has order p, φ can be extended to a homomorphism $\varphi^*\colon \langle x\rangle\to K$; but then

$$z=x^p\varphi=(x\varphi^*)^p,$$

so that z is a p-th power in K.

Thus in particular, if P is a finite p-group of exponent p^n, with $n>0$, there is a non-split extension of P by a cyclic group of order p^n; and if there is an element of $Z(P)$ which is not a p-th power in P, there is a non-split extension of P by a group of order p. This suggests the question: do there exist non-trivial finite p-groups for which all extensions by a group of order p split? We shall see in Theorem B that there are such p-groups, at least for $p\neq 3$.

First we give the general criterion for splitting of extensions by a group of order p.

Theorem A. *Let K be any group. All extensions of K by a group of order p split if and only if for each automorphism α of K and element a of K such that $a^\alpha = a$ and α^p is the inner automorphism of K induced by a, there is an element x of K such that*

$$a = x\, x^\alpha\, x^{\alpha^2} \dots x^{\alpha^{p-1}}.$$

Proof. Suppose first that $\alpha \in \operatorname{Aut} K$ and $a \in K$ with $a^\alpha = a$ and $k^{\alpha^p} = k^a$ for all $k \in K$. Let $\langle y \rangle$ be a cyclic group whose order is p times the order of a (infinite if the order of a is infinite). Then we can form a semi-direct product H of K by $\langle y \rangle$ with action defined by

$$k^y = k^\alpha \quad \text{for all } k \in K.$$

In H, $a^y = a$ and $k^{y^p} = k^a$ for all $k \in K$; therefore $y^p a^{-1} \in Z(H)$. Let $\bar{H} = H/\langle y^p a^{-1} \rangle$ and for each element or subset X of H, let $\bar{X}$ denote the natural image of X in $\bar{H}$. Then $\bar{K}$ is a normal subgroup of $\bar{H}$, $\bar{H} = \langle \bar{y} \rangle \bar{K}$, $\bar{y} \notin \bar{K}$ and $\bar{y}^p \in \bar{K}$. Therefore $\bar{K}$ has index p in $\bar{H}$. Moreover, in H, $\langle y^p a^{-1} \rangle \cap K = 1$, so that $\bar{K} \cong K$. Thus if all extensions of K by a group of order p split, $\bar{H}$ must split over $\bar{K}$.

If conversely $\bar{H}$ splits over $\bar{K}$ for all suitable choices of α and a, then all extensions of K by a group of order p split: for let G be any group containing K as a normal subgroup of index p. Choose $g \in G \setminus K$, so that

$$G = \langle g \rangle K.$$

Let $\langle y \rangle$ be a cyclic group of the same order as $\langle g \rangle$, and construct the semi-direct product of K by $\langle y \rangle$ with the action of $\langle y \rangle$ on K defined by

$$k^y = k^g \quad \text{for all } k \in K;$$

call this semi-direct product H. Then the map

$$\psi\colon\ y^r k \mapsto g^r k$$

(for all integers r and elements k of K) is a homomorphism of H onto G, and

$$\operatorname{Ker} \psi = \{ y^r k \mid g^r = k^{-1} \text{ in } G \}.$$

Since $g^p \in K$, we see that $y^p g^{-p} \in \operatorname{Ker} \psi$; and

$$(g^p)^y = (g^p)^g = g^p,$$

so that g^p and y commute in H. Then since $g^r \in K$ if and only if r is a multiple of p,

$$\operatorname{Ker} \psi = \langle y^p g^{-p} \rangle.$$

Let α be the automorphism of K defined by conjugation by g, and let $a = g^p$. Since $\operatorname{Ker} \psi$ is normal in H,

$$(y^p a^{-1})^y \in \operatorname{Ker} \psi,$$

that is

$$y^p a^{-\alpha} \in \operatorname{Ker} \psi.$$

It follows that $a^\alpha = a$ (and in fact that $\operatorname{Ker} \psi \leqq Z(H)$, since $y^p a^{-1}$ in any case centralizes K). Therefore α and a satisfy the hypotheses of the theorem, H is defined as in the first paragraph of the proof, and $\operatorname{Ker} \psi = \langle y^p a^{-1} \rangle$, so that ψ induces an isomorphism of $\overline{H}$ onto G in which $\overline{K}$ is mapped to K. Hence if $\overline{H}$ splits over $\overline{K}$, G splits over K.

For a particular α and a, $\overline{H}$ splits over $\overline{K}$ if and only if there is an element of order p in $\overline{H} \setminus \overline{K}$. Such an element must be of the form

$$\bar{y}^r \bar{x}, \quad \text{where } r \not\equiv 0 \ (p) \text{ and } x \in K.$$

But then there is an integer s such that $r s \equiv 1 \ (p)$, the element $(\bar{y}^r \bar{x})^s$ of $\overline{H}$ also has order p, and

$$(\bar{y}^r \bar{x})^s \overline{K} = \bar{y}^{rs} \overline{K} = \bar{y} \overline{K},$$

hence $(\bar{y}^r \bar{x})^s = \bar{y} \overline{x'}$ with $x' \in K$. Therefore we may as well assume that $r = 1$ above. Hence $\overline{H}$ splits over $\overline{K}$ if and only if there is an element $x \in K$ such that $(y x)^p \in \langle y^p a^{-1} \rangle$. By induction on m, for each positive integer m

$$(y x)^m = y^m x^{\alpha^{m-1}} x^{\alpha^{m-2}} \dots x^\alpha x.$$

Then since y and a commute,

$$(y x)^p \in \langle y^p a^{-1} \rangle \quad \text{if and only if } x^{\alpha^{p-1}} \dots x^\alpha x = a^{-1}.$$

Therefore (replacing x by x^{-1}) $\overline{H}$ splits over $\overline{K}$ if and only if there is an element $x \in K$ such that

$$a = x x^\alpha \dots x^{\alpha^{p-1}}.$$

This completes the proof.

If we want to show that a particular group K has a non-split extension by a group of order p, then according to Theorem A we must produce a suitable automorphism α of K and element a of K. When can α be chosen as an inner automorphism of K? If say α is conjugation by the element c of K, then α^p is conjugation by c^p and therefore a must be chosen as an element $z c^p$ with $z \in Z(K)$. Then

$$a^\alpha = (z c^p)^c = a.$$

For any $x \in K$,

$$\begin{aligned} x x^\alpha x^{\alpha^2} \dots x^{\alpha^{p-1}} &= x (c^{-1} x c)(c^{-2} x c^2) \dots (c^{-(p-1)} x c^{p-1}) \\ &= (x c^{-1})^p c^p. \end{aligned}$$

If z is a p-th power in K, say $z = b^p$, then by setting $x = b c \in K$ we find

$$x x^\alpha x^{\alpha^2} \dots x^{\alpha^{p-1}} = b^p c^p = z c^p = a.$$

Therefore with this choice of α, we can show that there is a non-split extension of K by a group of order p only if there is an element in $Z(K)$ which is not a p-th power in K: but this case is already covered by (III). So in general we are forced to consider outer automorphisms of K.

As an illustration we now prove

Theorem B. *Let K be a non-abelian group of order p^{n+1} (where $n \geqq 2$) with a cyclic subgroup of order p^n. Then there is a non-split extension of K by a group of order p, except that if either $p=2$, $n \geqq 3$ and K is quasi-dihedral or $p \geqq 5$ and $n=2$ then all extensions of K by a group of order p split.*

Remark. Every element of $Z(K)$ is a p-th power in K, so that (III) is not applicable.

Proof of Theorem B. The possible structures for K are well known: see for instance Huppert [1], I. 14.9, pp. 90–91. We consider them in turn.

(1) Suppose first that $p=2$. We observe directly that *if K is either dihedral or generalized quaternion then there is a non-split extension of K by a group of order* 2. This is clear, for if G is a generalized quaternion group of order 2^{n+2}, say $G=\langle x, y\rangle$ with $x^{2^{n+1}}=1$, $y^2=x^{2^n}$ and $x^y=x^{-1}$, then G has a subgroup $G_1=\langle x^2, y\rangle$ which is generalized quaternion of order 2^{n+1}; and since G has a unique element of order 2, G cannot split over G_1.

Again, if J is a quasi-dihedral group of order 2^{n+2}, say $J=\langle x, y\rangle$ with $x^{2^{n+1}}=1=y^2$ and $x^y=x^{2^n-1}$, then J has a subgroup $J_1=\langle x^2, y\rangle$ which is dihedral of order 2^{n+1}. An element $x^r y$ of J has order 2 if and only if $1=x^r(x^r)^y=x^{2^n r}$, that is if and only if r is even. The only other element of order 2 in J is x^{2^n}, so that in fact every element of order 2 in J lies in J_1 and therefore J cannot split over J_1.

(2) Now suppose that $p=2$ and $n \geqq 3$. There remain to be discussed two possibilities for K. In either case K is a semi-direct product of a cyclic group $\langle w\rangle$ of order 2^n by a group $\langle \eta\rangle$ of order 2. Since K is non-abelian the action of $\langle \eta\rangle$ on $\langle w\rangle$ is faithful and we may assume that $\eta \in \operatorname{Aut}\langle w\rangle$ and K is a subgroup of the holomorph of $\langle w\rangle$. It is well known (see Huppert [1], I. 13.19(c), p. 84) that

$$\operatorname{Aut}\langle w\rangle=\langle \gamma\rangle \times \langle \varepsilon\rangle,$$

where $w^\gamma=w^5$ and $w^\varepsilon=w^{-1}$. The elements of order 2 in $\operatorname{Aut}\langle w\rangle$ are $\gamma^{2^{n-3}}$, ε and $\gamma^{2^{n-3}}\varepsilon$. Then in K,

$$\eta=\text{either } \gamma^{2^{n-3}} \text{ or } \gamma^{2^{n-3}}\varepsilon$$

(for if $\eta=\varepsilon$ then K is dihedral, and we have disposed of this case already).

(3) Suppose that $\eta=\gamma^{2^{n-3}}$. We show that *then there is a non-split extension of K by a group of order* 2. This is clear if $n \geqq 4$, for then η is a square in $\operatorname{Aut}\langle w\rangle$ and we see that the subgroup $\langle w\rangle\langle \gamma^{2^{n-4}}\rangle$ of the holomorph of $\langle w\rangle$ does not split over $K=\langle w\rangle\langle \gamma^{2^{n-3}}\rangle$.

If $n=3$ we can apply Theorem A. It is straightforward to verify that there is an automorphism α of K such that

$$w^{\alpha}=w^{-1}\eta \quad \text{and} \quad \eta^{\alpha}=w^{4}\eta.$$

Then $\alpha^2=1$ and so, since $w^2\in Z(K)$, α^2 is the inner automorphism of K induced by w^2. Moreover $(w^2)^{\alpha}=w^2$. According to Theorem A it is now enough to show that there is no element $x\in K$ such that $w^2=x\,x^{\alpha}$. To establish this, we only need to verify that for $r=0, 1, 2, 3$,

$$\begin{aligned}(w^{2r})\,(w^{2r})^{\alpha}&=w^{4r},\\ (w^{2r+1})\,(w^{2r+1})^{\alpha}&=w^{4r}\eta,\\ (w^{2r}\eta)\,(w^{2r}\eta)^{\alpha}&=w^{4r+4},\\ (w^{2r+1}\eta)\,(w^{2r+1}\eta)^{\alpha}&=w^{4r}\eta.\end{aligned}$$

(4) Next suppose that in (2), $\eta=\gamma^{2^{n-3}}\varepsilon$. Then

$$w^{\eta}=\left(w^{(1+2^2)^{2^{n-3}}}\right)^{\varepsilon}=w^{-1+2^{n-1}},$$

so that K is quasi-dihedral. We show that *then all extensions of K by a group of order 2 split.*

Let $\alpha\in\operatorname{Aut}K$. If $w^{\alpha}=w^{s}\eta$ for some integer s then $(w^2)^{\alpha}=w^{s}(w^{s})^{\eta}=w^{2^{n-1}s}$: this is however impossible since the element $w^{2^{n-1}s}$ has order at most 2 whereas the element w^2 has order $2^{n-1}>2$, since $n\geqq 3$. Therefore $\langle w\rangle$ is in this case a characteristic subgroup of K, and

$$w^{\alpha}=w^{s} \quad \text{and} \quad \eta^{\alpha}=w^{t}\eta$$

for certain integers s and t. It is easy to check that the conditions imposed on s and t by the requirement that α be an automorphism of K are just

$$s\not\equiv 0 \ (2) \quad \text{and} \quad t\equiv 0 \ (2). \tag{i}$$

For the purpose of applying Theorem A, we now suppose that α^2 is inner. If α^2 were induced by an element of K of the form $w^{r}\eta$ then we should have

$$s^2\equiv 2^{n-1}-1 \ (2^n):$$

but this congruence has no solution for s, for it implies on the one hand that s^2+1 is divisible by $2^{n-1}(\geqq 4)$ and on the other that s is odd and hence that s^2+1 is twice an odd number.

Thus we may suppose that α^2 is the inner automorphism of K induced by w^{r} for some integer r. From this we obtain the conditions

$$s^2\equiv 1 \ (2^n) \quad \text{and} \quad (s+1)\,t\equiv(2^{n-1}-2)\,r \ (2^n). \tag{ii}$$

We require also that

$$(w^r)^\alpha = w^r,$$

which imposes the condition

$$(s-1)\, r \equiv 0 \ (2^n). \tag{iii}$$

Since $s-1$ and $s+1$ are consecutive even integers, it follows from (ii) that $s \pm 1$ is twice an odd number and $s \mp 1 \equiv 0 \ (2^{n-1})$. Therefore we may assume that

$$s = 1 \quad \text{or} \quad 2^{n-1}+1 \quad \text{or} \quad 2^{n-1}-1 \quad \text{or} \quad 2^n - 1. \tag{iv}$$

The assertion will follow from Theorem A if we show that for every choice of r, s and t satisfying (i), (ii), (iii) and (iv), there is an $x \in K$ such that

$$w^r = x\, x^\alpha.$$

Every element of K is expressible as either w^m or $w^m \eta$, and

$$w^m (w^m)^\alpha = w^{(s+1)\, m},$$
$$(w^m \eta)\, (w^m \eta)^\alpha = w^{m + (ms+t)\,(2^{n-1}-1)}.$$

Therefore we must show that for every compatible choice of r, s and t, one or other of the following congruences has a solution for m:

either

$$(s+1)\, m \equiv r \ (2^n)$$

or

$$(1 + 2^{n-1} s - s)\, m + (2^{n-1} - 1)\, t \equiv r \ (2^n). \tag{v}$$

We consider in turn the four possibilities for s, recalling that for integers a, b and d with $d > 0$, the linear congruence

$$a\, x \equiv b \ (d)$$

has an integral solution for x if and only if b is divisible by (a, d), the greatest common divisor of a and d.

If $s = 1$, the first congruence in (v) is

$$2m \equiv r \ (2^n).$$

This has a solution for m, since from (i) and (ii), r is even.

If $s = 2^{n-1} + 1$, the first congruence in (v) is

$$(2^{n-1} + 2)\, m \equiv r \ (2^n).$$

Since $n \geqq 3$, $(2^{n-1} + 2,\, 2^n) = 2$ and from (iii), r is even: therefore there is a solution for m.

If $s=2^{n-1}-1$, the first congruence in (v) is

$$2^{n-1}m\equiv r\ (2^n).$$

In this case $(2^{n-1},2^n)=2^{n-1}$, and from (iii), $r\equiv 0\ (2^{n-1})$, so again there is a solution for m.

Finally, if $s=2^n-1$, the second congruence in (v) is

$$(2-2^{n-1})\,m\equiv r-(2^{n-1}-1)\,t\ (2^n).$$

Here $(2-2^{n-1},2^n)=2$ and from (iii), r is even, and from (i), t is even. Therefore there is a solution for m.

(5) Now we suppose $p>2$. Then the structure of K is uniquely determined: it is a semi-direct product of a cyclic group $\langle w\rangle$ of order p^n by a group $\langle\eta\rangle$ of order p, and

$$w^\eta=w^{p^{n-1}+1}.$$

As in (2), we may assume that $\eta\in\operatorname{Aut}\langle w\rangle$ and K is a subgroup of the holomorph of $\langle w\rangle$. Now $\operatorname{Aut}\langle w\rangle$ is cyclic of order $p^{n-1}(p-1)$ and therefore has a unique subgroup Y of order p. Moreover when $n\geqq 3$, every element of Y is a p-th power in $\operatorname{Aut}\langle w\rangle$, so that there is an automorphism ξ of $\langle w\rangle$ such that $\eta=\xi^p$: but then the subgroup $\langle w\rangle\langle\xi\rangle$ of the holomorph of $\langle w\rangle$ contains $K=\langle w\rangle\langle\eta\rangle$ as a subgroup of index p and cannot split over K. Hence *if $n\geqq 3$, there is a non-split extension of K by a group of order p.*

(6) We suppose finally that $p>2$ and $n=2$. Then K is a semi-direct product of a cyclic group $\langle w\rangle$ of order p^2 by a group $\langle\eta\rangle$ of order p, with

$$w^\eta=w^{p+1}.$$

It is clear that

$$Z(K)=\langle w^p\rangle,$$

and it is easy to check that, since p is odd, for any integers a and b,

$$(w^a\eta^b)^p=w^{pa}.$$

We wish to apply Theorem A and are therefore interested in automorphisms α of K for which α^p is inner. Then the restriction of α to $Z(K)$ is a p-automorphism of $Z(K)$ and therefore has a non-trivial fixed point in $Z(K)$. Since $Z(K)$ has order p, it follows that

$$(w^p)^\alpha=w^p.$$

Now suppose that

$$w^\alpha=w^a\eta^b\quad\text{and}\quad \eta^\alpha=w^c\eta^d,$$

where a, b, c, d are integers. Then

$$w^p=(w^p)^\alpha=(w^a\eta^b)^p=w^{pa}$$

and

$$1=(\eta^p)^\alpha=(w^c\eta^d)^p=w^{pc}.$$

Hence
$$a \equiv 1 \ (p) \quad \text{and} \quad c \equiv 0 \ (p). \tag{i}$$
Using these conditions on a and c, a straightforward calculation shows that since α is to preserve the relation $w^\eta = w^{p+1}$, d must satisfy the condition
$$d \equiv 1 \ (p). \tag{ii}$$

Hence there are integers b, e, f such that
$$w^\alpha = w^{1+pe}\eta^b \quad \text{and} \quad \eta^\alpha = w^{pf}\eta.$$

Moreover, these equations define an automorphism α of K for any choice of integers b, e, f.

Next, one shows by induction on m that for any positive integer m,
$$w^{\alpha^m} = w^{1+pme+p2^{-1}m(m-1)fb}\,\eta^{mb}$$
and
$$\eta^{\alpha^m} = w^{pmf}\eta. \tag{iii}$$
Therefore, since p is odd,
$$\alpha^p = 1.$$

Hence α^p is the inner automorphism induced by the element a of K if and only if $a \in Z(K)$. Since also α fixes every element of $Z(K)$, what remains to be considered is whether or not every element of $Z(K)$ is expressible in the form $x\,x^\alpha x^{\alpha^2} \dots x^{\alpha^{p-1}}$ with $x \in K$ (for each choice of b, e, f).

Consider an arbitrary element of K, say
$$x = w^s \eta^t.$$

From the equations (iii) we see that for each $m = 1, 2, \dots, p-1$,
$$x^{\alpha^m} = (w\,\eta^{mb})^s\, w^{pm(es+ft)+p2^{-1}m(m-1)fbs}\,\eta^t.$$
Since
$$\sum_{m=1}^{p-1} m = \tfrac{1}{2}p(p-1) \equiv 0 \ (p)$$
for p odd, and
$$\sum_{m=1}^{p-1} \tfrac{1}{2}m(m-1) = \tfrac{1}{6}p(p-1)(p-2) \begin{cases} \equiv 0 \ (p) & \text{for } p>3, \\ = 1 & \text{for } p=3, \end{cases}$$
we see that
$$x\,x^\alpha x^{\alpha^2} \dots x^{\alpha^{p-1}} = \begin{cases} w^s\eta^t(w\,\eta^b)^s\eta^t(w\,\eta^{2b})^s\eta^t \dots (w\,\eta^{(p-1)b})^s\eta^t & \text{for } p>3, \\ w^s\eta^t(w\,\eta^b)^s\eta^t(w\,\eta^{2b})^s\eta^t\,w^{3fbs} & \text{for } p=3. \end{cases}$$
Let
$$v = w^s\eta^t(w\,\eta^b)^s\eta^t(w\,\eta^{2b})^s\eta^t \dots (w\,\eta^{(p-1)b})^s\eta^t.$$

One shows by induction on s that

$$(w\eta^{mb})^s = w^{s-p2^{-1}s(s-1)mb}\,\eta^{mbs},$$

so that since

$$\sum_{m=1}^{p-1} m \equiv 0\ (p),$$

$$v = w^s\eta^t\, w^s\eta^{bs+t}\, w^s\eta^{2bs+t} \dots w^s\eta^{(p-1)bs+t}.$$

Since for any integers r and s,

$$\eta^r w^s = w^{(1-pr)s}\,\eta^r,$$

we see that

$$v = w^S\eta^T,$$

where

$$S = s + (1-pt)\,s + \big(1-p(t+(bs+t))\big)\,s + \cdots + \left(1-p\sum_{j=0}^{p-2}(jbs+t)\right) s$$

and

$$T = t + (bs+t) + (2bs+t) + \cdots + \big((p-1)\,bs+t\big).$$

Thus

$$\begin{aligned} S &= \left(1+\sum_{i=1}^{p-1}\left(1-p\sum_{j=0}^{i-1}(jbs+t)\right)\right) s \\ &= ps - p\sum_{i=1}^{p-1}\left(\tfrac{1}{2}i(i-1)\,bs+it\right)s \\ &= ps - \tfrac{1}{6}p^2(p-1)(p-2)\,bs^2 - \tfrac{1}{2}p^2(p-1)\,ts \end{aligned}$$

and $T = \frac{1}{2}p(p-1)\,bs + pt$. Hence

$$v = \begin{cases} w^{ps} & \text{for } p>3, \\ w^{3s(1-bs)} & \text{for } p=3, \end{cases}$$

and so

$$x\,x^{\alpha}\,x^{\alpha^2}\dots x^{\alpha^{p-1}} = \begin{cases} w^{ps} & \text{for } p>3, \\ w^{3s(1-bs+fb)} & \text{for } p=3. \end{cases}$$

Since $Z(K) = \langle w^p \rangle$, it is now clear that when $p>3$, every element of $Z(K)$ is expressible in the appropriate form: in fact $w^{ps} = (w^s)(w^s)^{\alpha}(w^s)^{\alpha^2}\dots(w^s)^{\alpha^{p-1}}$. *Then every extension of K by a group of order p splits.*

So suppose that $p=3$. We show that *then there is a non-split extension of K by a group of order* 3 by showing that there exist integers r, b, f such that the congruence

$$r \equiv s(1-bs+fb)\ \ (3)$$

has no solution for s. If for instance we choose $b=-1$ and $f=1$ then the right side of this congruence is s^2. Since -1 is a quadratic non-residue (mod 3), the congruence above has no solution for s if we choose also $r=-1$. This means in fact that in this case the equations

$$w^{\alpha}=w\eta^{-1} \quad \text{and} \quad \eta^{\alpha}=w^3\eta$$

define an automorphism α of K such that $\alpha^3=1$, $w^{-3}\in Z(K)$ and $(w^{-3})^{\alpha}=w^{-3}$, but there is no element $x\in K$ such that $w^{-3}=x x^{\alpha} x^{\alpha^2}$. This completes the proof.

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Dr. John S. Rose
Department of Pure Mathematics
The University
Newcastle upon Tyne, England

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By JOHN S. ROSE

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SPLITTING PROPERTIES OF GROUP EXTENSIONS

By JOHN S. ROSE

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In the investigation of a composite group G, it is often helpful to know whether G splits over a particular normal subgroup K. The Schur–Zassenhaus theorem furnishes perhaps the best known and most frequently applied sufficient condition for this to happen. The question of splitting is considered in this paper by looking for conditions on a group K which ensure that all groups G which contain K as a normal subgroup, with quotient G/K belonging to some suitable class of groups, split over K. For instance, it is shown that all groups containing K normally split over K if and only if K has trivial centre and the full automorphism group of K splits over the group of inner automorphisms of K (Theorem 2.7). A similar result gives a necessary and sufficient splitting condition related to the Schur–Zassenhaus theorem (Theorem 2.8).

The paper is arranged in three sections. Section 1 contains a brief account of familiar facts from extension theory, and some preliminary splitting criteria. Section 2 begins with a necessary condition for the splitting of the extensions of one given group by another, when the second group is presented in a particular way (Lemma 2.1). This, together with the criteria from § 1, provides the key to the main results. Section 3 gives examples of the splitting behaviour of extensions, especially of generalized dihedral and analogous groups. In particular, it is shown that, if n is an odd integer greater than 1 and D_{2n} denotes the (ordinary) dihedral group of order $2n$, then all extensions of D_{2n} split if at least one prime divisor of n is congruent to -1 modulo 4 (Corollary 3.9).

Notation and terminology are for the most part standard. The symbol p is always used to denote a prime number, and ϖ to denote a set of primes; ϖ' denotes the set of all primes not belonging to ϖ.

1. Preliminary splitting criteria

An *extension* of a group K by a group Q is a short exact sequence

$$1 \longrightarrow K \longrightarrow G \xrightarrow{\varphi} Q \longrightarrow 1$$

in which K is identified with a normal subgroup of G. (All the groups in an extension will in this paper be expressed multiplicatively.) The extension is said to *split* if there is a homomorphism $\psi: Q \to G$ such that $\psi\varphi$ is the identity map on Q. It is well known and very easy to show that this happens if and only if there is a *complement* to K in G, that is a subgroup Q_1 of G such that $G = KQ_1$ and $K \cap Q_1 = 1$. Thus the splitting of the extension depends only on the structure of the group G, and not on the homomorphism φ in the extension. When the extension does split, one also says that *G splits over K*.

The special importance of split extensions rests on the fact that they can be constructed 'externally' by means of semi-direct products. For any groups K and Q, and a homomorphism $\rho: Q \to \operatorname{Aut} K$ (where $\operatorname{Aut} K$ denotes the group of all automorphisms of K), the set of formal products $\{yk: y \in Q,\ k \in K\}$ acquires the structure of a group G when multiplication is defined, for any elements y, y' of Q and k, k' of K, by

$$(yk)(y'k') = yy'k^{y'\rho}k'.$$

Then G is *the semi-direct product of K by Q with action ρ*; K is identified with a normal subgroup of G by identification of the elements k and $1k$, similarly Q is identified with a subgroup of G by identification of y and $y1$, and then juxtaposition of elements may be interpreted as multiplication. This semi-direct product determines a split extension of K by Q:

$$1 \longrightarrow K \longrightarrow G \xrightarrow{\pi} Q \longrightarrow 1,$$

where π denotes the homomorphism $yk \mapsto y$ of G onto Q.

Two extensions of K by Q,

$$1 \longrightarrow K \longrightarrow G \xrightarrow{\varphi} Q \longrightarrow 1 \quad \text{and} \quad 1 \longrightarrow K \longrightarrow G^* \xrightarrow{\varphi^*} Q \longrightarrow 1,$$

are called *equivalent* (or *congruent*) if there is a homomorphism $\gamma: G \to G^*$ making the following diagram commutative:

$$\begin{tikzcd}
 & & G \arrow[dd, "\gamma"'] \arrow[dr, "\varphi"] & & \\
1 \arrow[r] & K \arrow[ur] \arrow[dr] & & Q \arrow[r] & 1. \\
 & & G^* \arrow[ur, "\varphi^*"'] & &
\end{tikzcd}$$

It is easy to show that any split extension of K by Q is equivalent to one determined by a semi-direct product of K by Q.

The group of all inner automorphisms of K is denoted by $\operatorname{Inn} K$, and the group $\operatorname{Aut} K/\operatorname{Inn} K$ of automorphism classes of K by $\operatorname{Out} K$. Each extension of K by Q determines a homomorphism $\theta: Q \to \operatorname{Out} K$ which, in the terminology of P. Hall (see Gruenberg [**7**]), is called the *coupling* of the extension. If the extension in question is

$$1 \longrightarrow K \longrightarrow G \xrightarrow{\varphi} Q \longrightarrow 1,$$

θ is defined as follows. For each y in Q, choose an element g in G such that $g\varphi = y$, and let $y\theta$ be the residue class modulo $\operatorname{Inn} K$ of the automorphism of K induced by conjugation in G by g; $y\theta$ is independent of the choice of the element g. Equivalent extensions have the same coupling. (MacLane ([**10**]) uses the term *conjugation class* for θ instead of 'coupling'.)

An immediate consequence of the definitions is

LEMMA 1.1. *Let K and Q be groups, and ρ a homomorphism of Q into* $\operatorname{Aut} K$. *Let G denote the semi-direct product of K by Q with action ρ. The corresponding split extension of K by Q has coupling $\rho\nu$, where ν denotes the natural homomorphism of* $\operatorname{Aut} K$ *onto* $\operatorname{Out} K$.

Any homomorphism of Q into $\operatorname{Out} K$ will be called a *coupling of Q to K*. The fundamental problem of extension theory is to determine, for given groups K and Q, and a coupling θ of Q to K, the equivalence classes of extensions of K by Q with coupling θ. A solution in terms of cohomology groups was supplied by Eilenberg and MacLane ([**4**]); see also MacLane ([**10**] Chapter IV, § 8) and Gruenberg ([**7**]). The centre of K is denoted by $Z(K)$. Then θ induces naturally a homomorphism $Q \to \operatorname{Aut} Z(K)$: this defines the action of Q on $Z(K)$ in the cohomology groups which follow. There is at least one extension of K by Q with coupling θ if and only if a certain element of $H^3(Q, Z(K))$, denoted by $\operatorname{Obs}(Q, K, \theta)$, is zero. If $\operatorname{Obs}(Q, K, \theta) = 0$, then the equivalence classes of extensions of K by Q with coupling θ are in one-to-one correspondence with the elements of $H^2(Q, Z(K))$.

One would like to have criteria for the splitting of extensions. In view of Lemma 1.1, it is a simple matter to give one such criterion. A coupling θ of Q to K will be said to *lift* to $\operatorname{Aut} K$ if there is a homomorphism $\rho: Q \to \operatorname{Aut} K$ such that $\theta = \rho\nu$, where ν denotes the natural homomorphism of $\operatorname{Aut} K$ onto $\operatorname{Out} K$.

LEMMA 1.2. *Let K and Q be groups, and θ a coupling of Q to K. There is a split extension of K by Q with coupling θ if and only if θ lifts to* $\operatorname{Aut} K$.

Proof. If θ lifts to $\operatorname{Aut} K$, then $\theta = \rho\nu$ for some homomorphism $\rho: Q \to \operatorname{Aut} K$ and the natural homomorphism $\nu: \operatorname{Aut} K \to \operatorname{Out} K$. Let G denote the semi-direct product of K by Q with action ρ. Then by 1.1 the corresponding split extension of K by Q has coupling θ.

If, conversely, there is a split extension of K by Q with coupling θ, the extension is equivalent to one determined by a semi-direct product of K

by Q, say with action ρ. Then since equivalent extensions have the same coupling, 1.1 shows that θ lifts to $\operatorname{Aut} K$.

There may be several non-equivalent extensions of K by Q with a specified coupling, so that in general Lemma 1.2 does not give a criterion strong enough to decide whether a particular extension splits. But in certain circumstances, there is at most one equivalence class of extensions of K by Q corresponding to each coupling of Q to K, and then the following corollary of Lemma 1.2 is applicable.

COROLLARY 1.3. *Let K and Q be groups such that for each coupling of Q to K there is at most one equivalence class of extensions of K by Q. Then an extension of K by Q splits if and only if its coupling lifts to* $\operatorname{Aut} K$.

REMARK 1.4. *There is just one equivalence class of extensions of K by Q for each coupling of Q to K if either* (i) $Z(K) = 1$ *or* (ii) $Z(K)$ *and Q are finite with* $(|Z(K)|, |Q|) = 1$.

In both cases, $H^n(Q, Z(K)) = 0$ for all $n > 0$, so that (i) and (ii) are immediate consequences of the theorems of Eilenberg and MacLane mentioned above; there is also an elementary proof of (i) in [**12**].

Corollary 1.3 makes it possible to reduce the question of splitting of all extensions of a group K by groups of a suitable class to the question of splitting of certain subgroups of $\operatorname{Aut} K$ over $\operatorname{Inn} K$. A class $\mathfrak{X}$ of groups is called Q-*closed* if every homomorphic image of every $\mathfrak{X}$-group is also an $\mathfrak{X}$-group.

LEMMA 1.5. *Let $\mathfrak{X}$ be a* Q*-closed class of groups. Suppose that K is a group such that, for each $\mathfrak{X}$-group X and coupling θ of X to K, there is just one equivalence class of extensions of K by X with coupling θ. Then all extensions of K by $\mathfrak{X}$-groups split if and only if for every $\mathfrak{X}$-subgroup $J/\operatorname{Inn} K$ of $\operatorname{Out} K$, J splits over* $\operatorname{Inn} K$.

Proof. Suppose that, for every $\mathfrak{X}$-subgroup $J/\operatorname{Inn} K$ of $\operatorname{Out} K$, J splits over $\operatorname{Inn} K$. Let X be an $\mathfrak{X}$-group, and consider an extension of K by X, say with coupling θ. Let $\operatorname{Im}\theta = J/\operatorname{Inn} K$, an $\mathfrak{X}$-group since $\mathfrak{X}$ is Q-closed. Then, by hypothesis, the extension

$$1 \longrightarrow \operatorname{Inn} K \longrightarrow J \xrightarrow{\nu_1} \operatorname{Im}\theta \longrightarrow 1$$

splits, where ν_1 is defined by restriction of the natural homomorphism ν of $\operatorname{Aut} K$ onto $\operatorname{Out} K$. Then it is clear that θ lifts to $\operatorname{Aut} K$, and by 1.3 the extension with coupling θ splits.

Suppose conversely that all extensions of K by $\mathfrak{X}$-groups split. Let $X = J/\operatorname{Inn} K$, an $\mathfrak{X}$-subgroup of $\operatorname{Out} K$. Let $\iota: X \to \operatorname{Out} K$ be the inclusion map. By hypothesis, there is an extension of K by X with coupling ι, and this must split. Therefore, by 1.3, ι lifts to $\operatorname{Aut} K$, and it follows that J splits over $\operatorname{Inn} K$.

Remark 1.6. *If K is a group with trivial centre, then all extensions of K split if and only if* $\operatorname{Aut} K$ *splits over* $\operatorname{Inn} K$.

This follows from 1.4 and 1.5, but can also be seen directly by considering, for a group G containing K as a normal subgroup, the natural embedding of $G/C_G(K)$ in $\operatorname{Aut} K$. It will be shown in § 2 that the hypothesis that all extensions of K split implies that K has trivial centre; thus Remark 1.6 is subsumed in Theorem 2.7.

Bercov ([**3**]) proved that if N is a minimal normal subgroup of a finite group G, so that N is the direct product of conjugates of a simple group H, and if H is non-abelian and $\operatorname{Aut} H$ splits over $\operatorname{Inn} H$, then G splits over N. The hypotheses on H, in conjunction with a result of Fitting ([**5**] Satz 12) on the automorphism group of a direct product of copies of a non-abelian simple group, imply that N has trivial centre and $\operatorname{Aut} N$ splits over $\operatorname{Inn} N$. Thus Bercov's result follows from Remark 1.6. Bercov's proof is quite different, depending on the construction of an explicit complement to N in G in terms of a complement to $\operatorname{Inn} H$ in $\operatorname{Aut} H$.

Suppose that K and Q are groups such that, for each coupling of Q to K, there is just one equivalence class of extensions of K by Q. Consider a particular extension of K by Q; suppose that its coupling is θ, and let $\operatorname{Im}\theta = J/\operatorname{Inn} K$. The first part of the proof of Lemma 1.5 serves to show that if J splits over $\operatorname{Inn} K$ then the extension with coupling θ splits. But it is not in general true conversely that if the extension with coupling θ splits then J splits over $\operatorname{Inn} K$. For instance, if K is dihedral of order 10, then K has trivial centre. It is known that $\operatorname{Aut} K$ is isomorphic to the holomorph of a cyclic group of order 5 (see § 3). Therefore $\operatorname{Aut} K$ cannot split over $\operatorname{Inn} K$, since a cyclic group of order 4 does not split over its subgroup of order 2. Let Q be a cyclic group of order 4, and let ρ be an isomorphism of Q into $\operatorname{Aut} K$. Then the semi-direct product of K by Q with action ρ defines a split extension of K by Q with coupling $\rho\nu$, in the notation of Lemma 1.1; and $\operatorname{Im}(\rho\nu) = \operatorname{Aut} K/\operatorname{Inn} K$, but $\operatorname{Aut} K$ does not split over $\operatorname{Inn} K$.

2. Some necessary and sufficient splitting conditions

What restrictions are imposed on a group K by the condition that all extensions of K by groups of some suitable class split? Various answers to this question for various suitable classes can be found by means of

LEMMA 2.1. *Let K and X be groups and Y a central subgroup of X. Suppose that all extensions of K by X/Y split. Then any isomorphism of Y onto a central subgroup of K can be extended to a homomorphism of X into K.*

Proof. Suppose that μ is an isomorphism of Y onto the central subgroup L of K. Let $\lambda = \mu^{-1}$, so that λ is an isomorphism of L onto Y, and let

$$G = K \times X, \quad \text{the direct product,}$$

and

$$N = \{zz^{\lambda} \colon z \in L\}, \quad \text{a central subgroup of } G.$$

For each subgroup H of G, let $\bar{H} = HN/N$. Since $K \cap N = 1$, $\bar{K}$ is a normal subgroup of $\bar{G}$ isomorphic to K. Moreover $KN = KY$, so that

$$\bar{G}/\bar{K} \cong KX/KY \cong X/Y.$$

Therefore, by hypothesis, $\bar{G}$ splits over $\bar{K}$. Let $\bar{J}$ be a complement to $\bar{K}$ in $\bar{G}$, where $\bar{J} = J/N$ and $N \leqslant J \leqslant G$. Then

$$G = J(KN) = JK$$

and

$$J \cap K = (J \cap KN) \cap K = N \cap K = 1;$$

that is, J is a complement to K in G. It follows that for each x in X there is one and only one element k_x in K such that $k_x x \in J$. Since J is a subgroup of G, the map $x \mapsto k_x$ is a homomorphism, η say, of X into K. Then

$$J = \{x^{\eta}x \colon x \in X\}.$$

Since $N \leqslant J$, it follows that for each z in L there is an element x in X for which $z = x^{\eta}$ and $z^{\lambda} = x$. Hence

$$z^{\lambda\eta} = z = z^{\lambda\mu}, \quad \text{for every } z \text{ in } L.$$

Since $L^{\lambda} = Y$, this shows that η extends μ.

COROLLARY 2.2. *Let K be a finite group and let n be the largest integer such that K contains an element of order p^n. If all extensions of K by a cyclic group of order p^n split, then $|Z(K)|$ is not divisible by p.*

Proof. Suppose to the contrary that p divides $|Z(K)|$. Then $Z(K)$ has a subgroup L of order p. Let X be a cyclic group of order p^{n+1}. There is an isomorphism μ of the subgroup Y of X of order p onto L. Since X/Y is cyclic of order p^n, Lemma 2.1 shows that μ can be extended to a homomorphism η of X into K. Then

$$Y \cap \operatorname{Ker} \eta = \operatorname{Ker} \mu = 1.$$

This implies that $\operatorname{Ker} \eta = 1$, and hence that X can be embedded in K. But this contradicts the definition of n. Therefore p cannot divide $|Z(K)|$.

COROLLARY 2.3. *Let K be a finite group. Then the following statements are equivalent.*

(i) *All extensions of K split.*

(ii) *All extensions of K by finite groups split.*

(iii) $Z(K) = 1$ *and* $\operatorname{Aut} K$ *splits over* $\operatorname{Inn} K$.

Proof. Trivially, (i) implies (ii).

If (ii) is true, then by Corollary 2.2 there is no prime p which can divide $|Z(K)|$. Therefore $Z(K) = 1$. Then $\operatorname{Inn} K$ is naturally isomorphic to K, and, since $\operatorname{Aut} K/\operatorname{Inn} K$ is finite, (ii) also implies that $\operatorname{Aut} K$ splits over $\operatorname{Inn} K$. Thus (iii) is then true.

It has been pointed out in Remark 1.6 that (iii) implies (i).

For an infinite group K, the condition that all extensions of K by finite groups split is not in general sufficient to imply that $Z(K) = 1$. Consider for instance $K \cong \mathbf{Q}^+$, the additive group of rationals. Since K is abelian, $\operatorname{Out} K = \operatorname{Aut} K$ so that every homomorphism θ of a group X into $\operatorname{Out} K$ defines an action of X on K and hence a split extension of K by X with coupling θ. However, for any finite group X and any action of X on $\mathbf{Q}^+$, $H^n(X, \mathbf{Q}^+) = 0$ for every positive integer n (MacLane **[10]** 117, Corollary 5.4). Therefore all extensions of K by finite groups split.

Lemma 2.1 does yield some information here.

COROLLARY 2.4. *Suppose that K is a group such that all extensions of K by finite non-abelian simple groups split. Then either $Z(K)$ is torsion-free or K contains copies of all finite groups. In particular, if K is finite then* $Z(K) = 1$.

Proof. Suppose that $Z(K)$ is not torsion-free. Then $Z(K)$ contains a subgroup L of prime order, say p. In order to show that K contains copies of all finite groups, it is enough to show that there is a prime q such that for every positive integer n the special linear group $\mathrm{SL}(n, q)$ of degree n over the Galois field $\mathrm{GF}(q)$ can be embedded in K. By Dirichlet's theorem there are infinitely many primes congruent to 1 modulo p; choose q to be one of these (and if $p = 2$, choose $q > 3$). Then, say,

$$q - 1 = ps.$$

For any given positive integer n, choose t to be an integer relatively prime to s and such that $pt \geqslant n$. Since $\mathrm{SL}(n, q)$ can be embedded in $\mathrm{SL}(pt, q)$, it is enough to show that the latter group can be embedded in K. Let $X = \mathrm{SL}(pt, q)$ and let $Y = Z(X)$. Then the order of Y is the greatest common divisor of pt and $q - 1$; thus $|Y| = p$. There is an isomorphism of Y onto L, and since $X/Y = \mathrm{PSL}(pt, q)$, a finite non-abelian simple group, Lemma 2.1 shows that this isomorphism extends to a homomorphism η of X into K. Then

$$Y \cap \operatorname{Ker} \eta = 1.$$

However, Y is the only non-trivial normal subgroup of X (see Artin [**1**] 165, Theorem 4.9). Hence $\operatorname{Ker} \eta = 1$ and X can be embedded in K.

It would be interesting to know whether $Z(K)$ must be torsion-free when all extensions of K by finite groups split.

It can happen that all extensions of a finite group K by non-abelian simple groups split but not all extensions of K split. For example, suppose again that K is dihedral of order 10. Then K has trivial centre, $|\operatorname{Out} K| = 2$ and $\operatorname{Aut} K$ does not split over $\operatorname{Inn} K$. Thus there is an extension of K by a group of order 2 which does not split; but by Lemma 1.5 any extension of K by any group which does not have a quotient of order 2 splits.

If all extensions of K split, then in fact $Z(K) = 1$ even when K is infinite. In order to deduce this from Lemma 2.1, the following observation is helpful.

LEMMA 2.5. *For any cyclic group L of prime order or infinite order and any infinite set Λ, there is a group X which is nilpotent of class 2 and such that $Z(X) \cong L$ and $|X| = |\Lambda|$.*

Proof. Choose N to be a countable nilpotent group of class 2 such that $Z(N) \cong L$. For instance, N can be the group of all 3×3 unitriangular matrices, with coefficients in $\mathrm{GF}(p)$ if $|L| = p$, prime, or coefficients in $\mathbf{Z}$ if L is infinite.

Now let $D = \operatorname{Dr} N^{\Lambda}$, the direct power. This is also nilpotent of class 2 and

$$|D| = |\Lambda| = |D/Z(D)|.$$

Let y be a generator of $Z(N)$, and for each λ in Λ define f_λ in D by

$$f_\lambda(\nu) = \begin{cases} y & \text{if } \nu = \lambda, \\ 1 & \text{if } \nu \neq \lambda. \end{cases}$$

Then $f_\lambda \in Z(D)$, and

$$Z(D) = \langle f_\lambda : \lambda \in \Lambda \rangle.$$

Now let $C = \langle f_\lambda f_\mu^{-1} : \lambda, \mu \in \Lambda \rangle \leqslant Z(D)$, and let $X = D/C$. Then X is nilpotent and $|X| = |\Lambda|$. Certainly $Z(X) \geqslant Z(D)/C$. In fact, equality holds: for suppose that $f \in D$ and $fC \in Z(X)$. Then

$$[f, g] \in C \quad \text{for all elements } g \text{ of } D.$$

Choose g to be defined by

$$g(\nu) = \begin{cases} x & \text{if } \nu = \lambda, \\ 1 & \text{if } \nu \neq \lambda, \end{cases}$$

where x is an arbitrary element of N and λ is an arbitrary element of Λ. Then

$$[f,g](\nu) = [f(\nu), g(\nu)] = \begin{cases} [f(\lambda), x] & \text{if } \nu = \lambda, \\ 1 & \text{if } \nu \neq \lambda. \end{cases}$$

No element of C has support consisting of one point only; so the fact that $[f,g] \in C$ implies that

$$[f(\lambda), x] = 1.$$

Since this is true for any x in N, $f(\lambda) \in Z(N)$; and since this is true for any λ in Λ, $f \in Z(D)$. Hence

$$\begin{aligned} Z(X) &= Z(D)/C \\ &= \langle f_\lambda C \rangle, \end{aligned}$$

where λ is any particular element of Λ. Therefore X has class 2 and

$$Z(X) \cong Z(N) \cong L.$$

COROLLARY 2.6. *Suppose that K is a group such that all extensions of K by abelian groups split. Then $Z(K) = 1$.*

Proof. Assume to the contrary that $Z(K) \neq 1$. Then $Z(K)$ has a cyclic subgroup L of either prime order or infinite order. By Lemma 2.5 there is a group X which is nilpotent of class 2 and such that $Z(X) \cong L$ and $|X| > |K|$. Since $X/Z(X)$ is abelian, Lemma 2.1 shows that an isomorphism of $Z(X)$ onto L can be extended to a homomorphism η of X into K. Then

$$Z(X) \cap \operatorname{Ker} \eta = 1.$$

But since X is nilpotent, every non-trivial normal subgroup of X has non-trivial intersection with $Z(X)$. It follows that $\operatorname{Ker} \eta = 1$ and X can be embedded in K, in contradiction to $|X| > |K|$. Therefore $Z(K) = 1$.

An immediate consequence is

THEOREM 2.7. *The following two statements for a group K are equivalent.*

(i) *All extensions of K split.*

(ii) $Z(K) = 1$ *and* $\operatorname{Aut} K$ *splits over* $\operatorname{Inn} K$.

There is a related result of Baer ([**2**] Theorem 1) showing that a group K is a direct factor of every group containing it as a normal subgroup if and only if $Z(K) = 1$ and $\operatorname{Aut} K = \operatorname{Inn} K$.

Note. I am indebted to Professor F. Loonstra for bringing to my attention his paper [**9**]. Theorem 2.7 follows from Theorems 4.3 and 5.1 of that paper, by arguments different from those used here.

THEOREM 2.8. *Let K be a finite group and let $\mathfrak{X}$ be a* Q*-closed class of finite ϖ-groups which includes all cyclic p-subgroups of K for all p in ϖ. Then all extensions of K by $\mathfrak{X}$-groups split if and only if $Z(K)$ is a ϖ'-group and for every $\mathfrak{X}$-subgroup $J/\operatorname{Inn} K$ of* $\operatorname{Out} K$, *J splits over* $\operatorname{Inn} K$.

Proof. If $Z(K)$ is a ϖ'-group then, by 1.4, for any finite ϖ-group X and any coupling of X to K there is just one equivalence class of extensions of K by X. Therefore the whole result follows from Lemma 1.5 provided that $Z(K)$ is a ϖ'-group when all extensions of K by $\mathfrak{X}$-groups split. This is true by Corollary 2.2, since then all extensions of K by cyclic p-groups of the appropriate orders split, for all p in ϖ.

The Schur–Zassenhaus theorem gives as a sufficient condition for all extensions of a finite group K by finite ϖ-groups to split, that K be a ϖ'-group. Theorem 2.8 gives, when $\mathfrak{X}$ is the class of all finite ϖ-groups, a necessary and sufficient condition for all extensions of a finite group K by finite ϖ-groups to split. However, the Schur–Zassenhaus theorem is not an immediate consequence of Theorem 2.8: one can follow the usual inductive proof from the case of an abelian K (Schur's theorem), which does come at once from Theorem 2.8. The proof of Schur's theorem contained in Theorem 2.8 is in the spirit of the proof given by Huppert ([**8**] 122, I.17.5).

COROLLARY 2.9. *Let K and $\mathfrak{X}$ be as in Theorem* 2.8, *and suppose that $Z(K)$ is a ϖ'-group. If for some term Z_i of the upper central series of K all extensions of K/Z_i by $\mathfrak{X}$-groups split, then all extensions of K by $\mathfrak{X}$-groups split.*

Proof. Let the upper central series of K be

$$1 = Z_0 \leqslant Z_1 \leqslant Z_2 \leqslant \dots.$$

The fact that $Z(K) = Z_1$ is a ϖ'-group implies that Z_i is a ϖ'-group (Huppert [**8**] 266, III. 2.13). Also if L/Z_1 is a cyclic p-subgroup of K/Z_1 for some $p \in \varpi$, then since Z_1 is a ϖ'-group L splits over Z_1, hence K has a cyclic p-subgroup of the same order as L/Z_1. Therefore it is enough for the proof to assume $i = 1$. Suppose then that all extensions of K/Z_1 by $\mathfrak{X}$-groups split. To show that all extensions of K by $\mathfrak{X}$-groups split it is enough, by Theorem 2.8 and since $Z(K)$ is a ϖ'-group, to show that, for any $\mathfrak{X}$-subgroup $J/\operatorname{Inn} K$ of $\operatorname{Out} K$, J splits over $\operatorname{Inn} K$. But since $\operatorname{Inn} K \simeq K/Z_1$ this is true by the supposition.

COROLLARY 2.10. *Let K and $\mathfrak{X}$ be as in Theorem* 2.8 *and suppose that K is nilpotent. All extensions of K by $\mathfrak{X}$-groups split if and only if K is a ϖ'-group.*

Proof. If K is a ϖ'-group then Corollary 2.9 shows that all extensions of K by $\mathfrak{X}$-groups split. Conversely, if all extensions of K by $\mathfrak{X}$-groups split then, by Theorem 2.8, $Z(K)$ is a ϖ'-group. Since K is nilpotent this implies that K is a ϖ'-group (Huppert [**8**] 266, III.2.13).

Every finite group G has a unique perfect normal subgroup K with soluble quotient G/K: K is the 'perfect radical' of G and G/K is the 'soluble residual' of G. One would like to know when G splits over K. Some information about this question is provided by Theorem 2.8 when ϖ is chosen to be the set of all primes and $\mathfrak{X}$ the class of all finite soluble groups.

COROLLARY 2.11. *Let K be a finite perfect group. Every finite group G with perfect radical K splits over K if and only if $Z(K) = 1$ and for every soluble subgroup $J/\operatorname{Inn} K$ of* $\operatorname{Out} K$, *J splits over* $\operatorname{Inn} K$.

3. Extensions of generalized dihedral and analogous groups

Let A be any abelian group other than an elementary 2-group, and let η denote the automorphism $a \mapsto a^{-1}$ of A. The semi-direct product of A by the group $\langle \eta \rangle$ of order 2 with the natural action is a non-abelian group denoted by $\operatorname{Dih} A$, and called a *generalized dihedral* group. The *holomorph* of any group G is denoted by $\operatorname{Hol} G$: it is the semi-direct product of G by $\operatorname{Aut} G$ with the natural action. It is known that $\operatorname{Aut}(\operatorname{Dih} A)$ is isomorphic to $\operatorname{Hol} A$ (Miller, Blichfeldt, Dickson [**11**] 169). In fact, A is characteristic in $\operatorname{Dih} A$ and the map

$$\psi\colon \operatorname{Aut}(\operatorname{Dih} A) \to \operatorname{Hol} A$$

defined by

$$\psi\colon \alpha \mapsto \alpha_1 a_\alpha,$$

where α_1 is the restriction of α to A and $\eta^\alpha = a_\alpha \eta$ is an isomorphism of $\operatorname{Aut}(\operatorname{Dih} A)$ onto $\operatorname{Hol} A$. Now

$$\operatorname{Inn}(\operatorname{Dih} A) = \langle \tau_\eta, \tau_a \colon a \in A \rangle,$$

where τ_a denotes conjugation by a and τ_η conjugation by η in $\operatorname{Dih} A$. Then, since in $\operatorname{Dih} A$,

$$\eta^{\tau_a} = a^{-1}\eta a = \eta a^2,$$

one sees that

$$\operatorname{Inn}(\operatorname{Dih} A) \xrightarrow{\psi} \langle \eta \rangle A^2,$$

where A^2 is the subgroup of A consisting of all squares of elements of A. In conjunction with Theorem 2.8 this enables one to prove that *if A is finite then every extension of* $\operatorname{Dih} A$ *by every group of odd order splits*. Dr P. M. Neumann pointed out to me, however, that this is very easily seen by an application of the Schur–Zassenhaus theorem and the Frattini argument, taken with the fact that the Sylow 2-subgroups of $\mathrm{Dih}\, A$ are self-normalizing. The argument may be formulated generally in the following way. A subgroup L has been called by Wielandt ([**14**]) *intravariant* in a group K if, for every α in $\mathrm{Aut}\, K$, L^α is a conjugate of L in K.

REMARK 3.1. *All extensions of the finite group K by finite ϖ-groups split if K has a self-normalizing intravariant ϖ'-subgroup L.*

To see this, suppose that K is a normal subgroup of a finite group G with G/K a ϖ-group. Since L is intravariant in K, $G = N_G(L)K$, and because L is self-normalizing in K, $N_G(L) \cap K = L$. Therefore

$$N_G(L)/L \cong G/K,$$

so that because L is a ϖ'-group, $N_G(L)$ splits over L by the Schur–Zassenhaus theorem. Then a complement to L in $N_G(L)$ is also a complement to K in G.

As an interesting particular case, note that, *if K is any finite soluble group and the Carter subgroups of K are ϖ'-subgroups, then all extensions of K by finite ϖ-groups split.*

The extensions of a finite generalized dihedral group $\mathrm{Dih}\, A$ seem to be predisposed to split. The question to be considered is for which choices of A all extensions of $\mathrm{Dih}\, A$ split. By Corollary 2.3 such a $\mathrm{Dih}\, A$ must have trivial centre, and therefore A must have odd order greater than 1.

In this connection it is perhaps worth while to consider a slightly wider class of finite metabelian groups with trivial centre. With this aim, the following notation is introduced and retained for the rest of the paper.

Notation. A denotes a finite abelian group with a fixed-point-free automorphism η of prime order q. (This implies that $|A| \equiv 1(q)$.) *Then E denotes the semi-direct product of A by $\langle \eta \rangle$ with the natural action.* Thus E is a subgroup of $\mathrm{Hol}\, A$.

These groups E include all finite generalized dihedral groups with trivial centre, for when $|A|$ is odd the automorphism $a \mapsto a^{-1}$ of A is fixed-point-free. In fact these are the only groups E which occur for $q = 2$.

LEMMA 3.2. *A is characteristic in E, every element of E outside A has order q, and $Z(E) = 1$.*

Proof. Since q does not divide $|A|$, A is the set of all q'-elements of E. Let x be an element of E outside A. Then $x = \eta^r a$, where $0 < r < q$ and $a \in A$. Since $x^q \in A$,

$$x^q = x^{-1}x^q x = a^{-1}\eta^{-r}x^q\eta^r a = \eta^{-r}x^q\eta^r,$$

so that $x^q \in C_A(\eta^r)$. But $\langle \eta^r \rangle = \langle \eta \rangle$ and so

$$C_A(\eta^r) = C_A(\eta) = 1, \quad \text{by hypothesis.}$$

Hence $x^q = 1$. Finally, $Z(E) \leqslant A$ (since otherwise A would be a direct factor of E) and therefore $Z(E) \leqslant C_A(\eta) = 1$.

It follows from the fact that $C_A(\eta) = 1$ that the Sylow q-subgroups of E are self-normalizing. Hence, by Remark 3.1, *every extension of E by every finite q'-group splits.* It is necessary in order to discuss further splitting properties of extensions of E to investigate $\operatorname{Aut} E$. My original proof of Lemma 3.3 below depended on an explicit description of the automorphisms of E. I am indebted to the referee for the suggestion of an appeal to a simple cohomological result in order to eliminate a certain amount of tedious computation.

LEMMA 3.3. *$\operatorname{Aut} E$ splits over $\operatorname{Inn} E$ if and only if $N_{\operatorname{Aut} A}(\langle \eta \rangle)$ splits over $\langle \eta \rangle$.*

Proof. It follows from Lemma 3.2 that if $\theta \in \operatorname{Aut} E$ then the restriction θ_1 of θ to A is an automorphism of A; and clearly $\eta^\theta = \eta^r a$, where $0 < r < q$ and $a \in A$. Since

$$\eta^{-1} b \eta = b^\eta \quad \text{for all } b \text{ in } A,$$

$$a^{-1}\eta^{-r} b^{\theta_1} \eta^r a = b^{\eta\theta_1},$$

that is

$$b^{\theta_1 \eta^r} = b^{\eta\theta_1}$$

for all b in A. Thus

$$\theta_1^{-1}\eta\theta_1 = \eta^r$$

so that

$$\theta_1 \in N_{\operatorname{Aut} A}(\langle \eta \rangle) = N, \quad \text{say.}$$

Hence the map $\theta \mapsto \theta_1$ defines a homomorphism of $\operatorname{Aut} E$ into N. This is in fact an epimorphism, because if $\alpha \in N$ then

$$\alpha^{-1}\eta\alpha = \eta^r \quad \text{for some } r,$$

and a straightforward verification shows that the map

$$\eta^m b \mapsto \eta^{rm} b^\alpha \quad (0 \leqslant m < q,\ b \in A)$$

is an automorphism α^* of E such that ${\alpha^*}_1 = \alpha$.

If $\theta \in \operatorname{Aut} E$ and $\theta_1 = 1$, then $\theta_1^{-1}\eta\theta_1 = \eta$ and so $\eta^\theta = \eta a$: hence θ fixes both A and E/A elementwise. But the group of all automorphisms of E which fix A and E/A elementwise is isomorphic to the group of all crossed homomorphisms of E/A to A, where A is viewed in the natural way as an $\langle \eta \rangle$-module (MacLane [**10**] 106, Proposition 2.2). Since $(|E/A|, |A|) = 1$, $H^1(E/A, A) = 0$ so that every crossed homomorphism is principal. Therefore (MacLane, loc. cit.) θ is an inner automorphism of E induced by an element of A. Since $Z(E) = 1$, the inner automorphisms of E induced by elements of A form a group isomorphic to A. Hence there is a short exact sequence

$$1 \to A \to \operatorname{Aut} E \to N \to 1.$$

Moreover, this extension of A by N splits: for the map $\alpha \mapsto \alpha^*$ is a homomorphism of N into $\operatorname{Aut} E$ such that $\alpha^*{}_1 = \alpha$.

Restriction of the sequence above clearly yields the exact sequence

$$1 \to A \to \operatorname{Inn} E \to \langle \eta \rangle \to 1.$$

Hence there is a commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & A \arrow[r] \arrow[d, equal] & \operatorname{Inn} E \arrow[r] \arrow[d] & \langle \eta \rangle \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & A \arrow[r] & \operatorname{Aut} E \arrow[r, shift left] \arrow[d] & N \arrow[r] \arrow[l, shift left] \arrow[d] & 1 \\
 & & \operatorname{Out} E \arrow[d] & N/\langle \eta \rangle \arrow[d] & \\
 & & 1 & 1 &
\end{tikzcd}$$

There is an induced homomorphism $\operatorname{Out} E \to N/\langle \eta \rangle$, preserving commutativity of the diagram; and this is seen from the diagram to be an isomorphism. Then it is clear that either one of the vertical extensions splits if and only if the other one also splits.

Since $Z(E) = 1$, and in view of Corollary 2.3, Lemma 3.3 already provides a necessary and sufficient condition for all extensions of E to split. A better condition will be given in Theorem 3.5. The following simple lemma is used in the proof.

LEMMA 3.4. *Let Q be a finite abelian q-group and let y be an element of Q of order q. Then $\langle y \rangle$ is a direct factor of Q if and only if y is not a qth power in Q.*

Proof. Let ν denote the endomorphism $x \mapsto x^q$ of Q. Then if $Q = \langle y \rangle \times Q_0$ for some subgroup Q_0 of Q, clearly $\operatorname{Im} \nu \leqslant Q_0$ and so y is not a qth power in Q. Conversely, suppose that y is not a qth power in Q. Let $Q_1 = \operatorname{Im} \nu$. Then Q/Q_1 is an elementary q-group and $y \notin Q_1$. Hence

$$Q/Q_1 = \langle yQ_1 \rangle \times Q_0/Q_1$$

for some subgroup Q_0 of index q in Q. Then $y \notin Q_0$ and therefore, because y has order q, $Q = \langle y \rangle \times Q_0$.

THEOREM 3.5. *Let Q be a Sylow q-subgroup of $C_{\mathrm{Aut}A}(\eta)$. Then all extensions of E split if and only if $\langle \eta \rangle$ is a direct factor of Q. If Q is abelian, then either there is an extension of E by a group of order q which does not split or all extensions of E split* (*according as η is or is not a qth power in Q*).

Proof. By Corollary 2.3, Lemma 3.2, and Lemma 3.3, all extensions of E split if and only if $N_{\mathrm{Aut}A}(\langle \eta \rangle)$ splits over $\langle \eta \rangle$. Since $|N_{\mathrm{Aut}A}(\langle \eta \rangle)/C_{\mathrm{Aut}A}(\eta)|$ is a divisor of $q-1$, Q is also a Sylow q-subgroup of $N_{\mathrm{Aut}A}(\langle \eta \rangle)$. Then, since $\langle \eta \rangle$ is an abelian normal q-subgroup of $N_{\mathrm{Aut}A}(\langle \eta \rangle)$, a theorem of Gaschütz ([**6**]) shows that $N_{\mathrm{Aut}A}(\langle \eta \rangle)$ splits over $\langle \eta \rangle$ if and only if Q splits over $\langle \eta \rangle$. But $\eta \in Z(Q)$ and so Q splits over $\langle \eta \rangle$ if and only if $\langle \eta \rangle$ is a direct factor of Q.

Now suppose that Q is abelian. If η is not a qth power in Q then, by Lemma 3.4, $\langle \eta \rangle$ is a direct factor of Q and therefore all extensions of E split. Suppose then that η is a qth power in Q, say $\eta = \alpha^q$ with $\alpha \in Q$. Then the semi-direct product G of A by $\langle \alpha \rangle$ with the natural action contains E as a normal subgroup of index q. Furthermore, G cannot split over E, for if it did the cyclic group G/A of order q^2 would split over its subgroup E/A of order q, which is false. Thus there is an extension of E by a group of order q which does not split.

Note. The last part of the argument above shows that in any case, if η is a qth power in $\mathrm{Aut}\,A$, then there is an extension of E by a group of order q which does not split. I do not know whether the condition that η is not a qth power in $\mathrm{Aut}\,\mathrm{A}$ is sufficient, without a condition on Q, to imply that all extensions of E split.

It is now quite easy to classify, as far as the splitting behaviour of their extensions is concerned, the groups E for which A is cyclic. The first point to observe is that if a finite cyclic group A has a fixed-point-free automorphism then $|A|$ is odd: for every automorphism of a cyclic group of even order fixes its unique element of order 2.

A cyclic group of odd order greater than 1 has in general many other fixed-point-free automorphisms besides the certain one, $a \mapsto a^{-1}$. Some of these may have prime orders.

LEMMA 3.6. *Let p be an odd prime and suppose that A is a cyclic group of order p^n, where $n \geqslant 1$. The fixed-point-free automorphisms of A are precisely those whose orders are not powers of p.*

Proof. Suppose $A = \langle a \rangle$. Let α be an automorphism of A which has a non-trivial fixed point. Then α must fix every element of the unique subgroup of A of order p, so that $(a^{p^{n-1}})^\alpha = a^{p^{n-1}}$. Let $a^\alpha = a^r$, where $0 < r < p^n$ and $r \not\equiv 0\ (p)$. Then $a^{p^{n-1}r} = a^{p^{n-1}}$, hence $p^{n-1}r \equiv p^{n-1}\ (p^n)$ and so $r \equiv 1\ (p)$. Then $r = 1+ps$ for some $s \geqslant 0$, so that

$$r^{p^{n-1}} = (1+ps)^{p^{n-1}} \equiv 1\ (p^n).$$

Hence

$$a^{\alpha^{p^{n-1}}} = a^{r^{p^{n-1}}} = a,$$

so that $\alpha^{p^{n-1}} = 1$. Therefore α has order a power of p. If, conversely, α is an automorphism of A of order a power of p, then $\langle\alpha\rangle$ is a p-group acting on the p-group A and therefore, by the usual counting of orbits, α has a non-trivial fixed point in A.

If now A is a cyclic group of odd order greater than 1, with say $|A| = \prod_{i=1}^{s} p_i^{n_i}$, where $p_1, \ldots, p_s$ are the distinct prime divisors of $|A|$, then

$$|\operatorname{Aut} A| = \prod_{i=1}^{s} (p_i^{n_i} - p_i^{n_i-1});$$

and it follows readily from Lemma 3.6 that the number of fixed-point-free automorphisms of A is

$$\prod_{i=1}^{s} (p_i^{n_i} - 2p_i^{n_i-1}).$$

Which of these have prime order q?

Lemma 3.7. *Let A be a cyclic group of odd order greater than* 1, *and let the distinct prime divisors of $|A|$ be $p_1, \ldots, p_s$ with, say, $|A| = \prod_{i=1}^{s} p_i^{n_i}$. If A has a fixed-point-free automorphism of prime order q, then $p_i \equiv 1\ (q)$ for $i = 1, \ldots, s$. Conversely, for each common prime divisor q of $p_1 - 1, \ldots, p_s - 1$, A has $(q-1)^s$ fixed-point-free automorphisms of order q.*

Proof. For each $i = 1, \ldots, s$, let A_i denote the Sylow p_i-subgroup of A, and for each $\alpha \in \operatorname{Aut} A$ let α_i denote the automorphism of A_i defined by restriction of α to A_i. Then the map

$$\alpha \mapsto (\alpha_1, \ldots, \alpha_s)$$

is an isomorphism of $\operatorname{Aut} A$ onto the (external) direct product $\operatorname{Aut} A_1 \times \ldots \times \operatorname{Aut} A_s$. Moreover, α is fixed-point-free of prime order q if and only if each α_i is fixed-point-free of order q $(i = 1, \ldots, s)$. Now $\operatorname{Aut} A_i$ is cyclic of order $p_i^{n_i-1}(p_i - 1)$, so that since q is prime Lemma 3.6 shows that A_i has a fixed-point-free automorphism of order q only if q divides $p_i - 1$; and further, if q does divide $p_i - 1$, that all the $q-1$ elements of $\operatorname{Aut} A_i$ of order q are fixed-point-free. The result follows.

THEOREM 3.8. *Suppose that the subgroup A of E is cyclic (and hence A has odd order greater than 1). Let the distinct prime divisors of $|A|$ be $p_1, \ldots, p_s$. Then $p_i \equiv 1\ (q)$ for $i = 1, \ldots, s$; and if in fact $p_i \equiv 1\ (q^2)$ for $i = 1, \ldots, s$ then there is an extension of E by a group of order q which does not split, but if, for some i, $p_i \not\equiv 1\ (q^2)$ then all extensions of E split.*

Proof. Since by hypothesis A has the fixed-point-free automorphism η of prime order q, Lemma 3.7 shows that $p_i \equiv 1\ (q)$ for $i = 1, \ldots, s$. Because in this case $\operatorname{Aut} A$ is abelian, the result will follow from Theorem 3.5 once it has been shown that η is a qth power in $\operatorname{Aut} A$ if and only if $p_i \equiv 1\ (q^2)$ for $i = 1, \ldots, s$.

Let A_i denote the Sylow p_i-subgroup of A and let η_i denote the automorphism of A_i defined by restriction of η to A_i, for $i = 1, \ldots, s$. Then η is a qth power in $\operatorname{Aut} A$ if and only if η_i is a qth power in $\operatorname{Aut} A_i$ for all i. Now $\operatorname{Aut} A_i$ is cyclic of order $p_i^{n_i-1}(p_i-1)$ and η_i has order q, so that η_i is a qth power in $\operatorname{Aut} A_i$ if and only if the order of the Sylow q-subgroup of $\operatorname{Aut} A_i$ is at least q^2, thus if and only if $p_i - 1$ is divisible by q^2. Hence η is a qth power in $\operatorname{Aut} A$ if and only if $p_i \equiv 1\ (q^2)$ for all i.

An interesting particular case is that of the ordinary dihedral groups, which are the groups $\operatorname{Dih} A$ with A cyclic.

COROLLARY 3.9. *Let n be an odd integer greater than 1, and let D_{2n} denote the dihedral group of order $2n$. If every prime divisor of n is congruent to 1 (mod 4) then there is an extension of D_{2n} by a group of order 2 which does not split, but if at least one prime divisor of n is congruent to -1 (mod 4) then all extensions of D_{2n} split.*

Among the groups E, as well as those for which A is cyclic the ones for which A is elementary are of special interest. It is convenient when A is an elementary p-group to suppose a base of A (regarded as a vector space over $\mathrm{GF}(p)$) fixed, and to identify each automorphism of A with the non-singular matrix over $\mathrm{GF}(p)$ which represents it with respect to this base. Then, if A has order p^n, $\operatorname{Aut} A$ is identified with $\mathrm{GL}(n, p)$. With this identification made it is clear that an automorphism α of A is fixed-point-free if and only if the matrix $\alpha - 1$ is non-singular.

The minimal polynomial of η over $\mathrm{GF}(p)$ must divide $x^q - 1$. Let t be the order of p (mod q), that is t is the least positive integer such that $p^t \equiv 1\ (q)$. (Then t divides both n and $q-1$.) The field $\mathrm{GF}(p^t)$ is the splitting field of $x^q - 1$ over $\mathrm{GF}(p)$, and $x^q - 1$ factorizes over $\mathrm{GF}(p^t)$ as a product of q distinct linear factors. Now η may be viewed as an element of $\mathrm{GL}(n, p^t)$. By the preceding argument, the minimal polynomial of η over $\mathrm{GF}(p^t)$ must be a product of distinct linear factors. Therefore by a standard result of vector space theory, η is conjugate in $\mathrm{GL}(n,p^t)$ to a diagonal matrix. The main result for elementary A can now be stated.

THEOREM 3.10. *Suppose that the subgroup A of E is elementary abelian of order p^n. Let t be the order of p (mod q), so that η is conjugate in $\mathrm{GL}(n,p^t)$ to a diagonal matrix. Let the distinct eigenvalues of η in $\mathrm{GF}(p^t)$ be $\lambda_1, \ldots, \lambda_r$ with multiplicities $l_1, \ldots, l_r$ respectively, where $\sum_{i=1}^{r} l_i = n$.*

(i) *If $p^t \not\equiv 1$ (q^2) and if for some i, l_i is not divisible by q, then all extensions of E split.*

(ii) *If either $p^t \equiv 1$ (q^2) or q divides l_i for all $i = 1, \ldots, r$, then there is an extension of E by a group of order q which does not split.*

COROLLARY 3.11. *Let A be an elementary abelian group of order p^n, where p is an odd prime and n a positive integer. If p is congruent to 1 (mod 4) or if n is even then there is an extension of* Dih A *by a group of order 2 which does not split, but if p is congruent to -1 (mod 4) and n is odd then all extensions of* Dih A *split.*

For the proof of Theorem 3.10, it is convenient to note

LEMMA 3.12. *If p, q are primes and t, n positive integers such that $p^t \equiv 1$ (q), $p^t \not\equiv 1$ (q^2) and q does not divide n, and if λ is an element of order q in $\mathrm{GF}(p^t)^{\times}$, then $\langle \lambda I_n \rangle$ is a direct factor of $\mathrm{GL}(n,p^t)$, where I_n denotes the $n \times n$ unit matrix.*

Proof. Let $\eta = \lambda I_n$. Then

$$\det \eta = \lambda^n \neq 1,$$

since n is not a multiple of q. Thus $\eta \notin \mathrm{SL}(n,p^t)$. Since $\mathrm{GL}(n,p^t)/\mathrm{SL}(n,p^t)$ is cyclic of order p^t-1, $\mathrm{GL}(n,p^t)$ has a normal subgroup U of index q containing $\mathrm{SL}(n,p^t)$. Since $p^t \not\equiv 1$ (q^2), q does not divide $|U/\mathrm{SL}(n,p^t)|$ and therefore, because η has order q, $\eta \notin U$. Hence $\langle \eta \rangle$ is a complement to U in $\mathrm{GL}(n,p^t)$. Since $\eta \in Z(\mathrm{GL}(n,p^t))$ it follows that $\langle \eta \rangle$ is a direct factor of $\mathrm{GL}(n,p^t)$.

Proof of Theorem 3.10. By hypothesis, η is conjugate in $\mathrm{GL}(n,p^t)$ to the blocked matrix

$$\tilde{\eta} = \begin{pmatrix} \lambda_1 I_{l_1} & 0 & \ldots & 0 \\ 0 & \lambda_2 I_{l_2} & \ldots & 0 \\ . & . & . & . \\ 0 & 0 & \ldots & \lambda_r I_{l_r} \end{pmatrix},$$

where for each positive integer m, I_m denotes the $m \times m$ unit matrix over $\mathrm{GF}(p^l)$. Simple calculations show that $C_{\mathrm{GL}(n,p^l)}(\tilde{\eta})$ consists of all blocked matrices

$$\begin{pmatrix} \alpha_1 & 0 & \dots & 0 \\ 0 & \alpha_2 & \dots & 0 \\ . & . & . & . \\ 0 & 0 & \dots & \alpha_r \end{pmatrix},$$

where $\alpha_i \in \mathrm{GL}(l_i, p^l)$ for $i = 1, \dots, r$. Thus there is an isomorphism of $C_{\mathrm{GL}(n,p^l)}(\tilde{\eta})$ onto the (external) direct product

$$\mathrm{GL}(l_1, p^l) \times \mathrm{GL}(l_2, p^l) \times \dots \times \mathrm{GL}(l_r, p^l) = G, \quad \text{say},$$

in which $\tilde{\eta}$ is mapped to the element

$$(\eta_1, \eta_2, \dots, \eta_r) = \hat{\eta}, \quad \text{say},$$

where $\eta_i = \lambda_i I_{l_i}$ for $i = 1, \dots, r$.

Now suppose that $p^l \not\equiv 1\ (q^2)$ and, for some i, l_i is not divisible by q: say q does not divide l_1. Since η is fixed-point-free, $\det(\eta - I_n) \neq 0$ and therefore $\det(\tilde{\eta} - I_n) \neq 0$. Hence $\lambda_i \neq 1$ for all $i = 1, \dots, r$. It follows that λ_1 has order q as an element of $\mathrm{GF}(p^l)^{\times}$, and therefore, by Lemma 3.12, $\langle \eta_1 \rangle$ is a direct factor of $\mathrm{GL}(l_1, p^l)$: say $\mathrm{GL}(l_1, p^l) = \langle \eta_1 \rangle \times U$.

For each $i = 1, \dots, r$, let π_i denote the projection homomorphism of G onto $\mathrm{GL}(l_i, p^l)$ and let σ_i denote the injection of $\mathrm{GL}(l_i, p^l)$ in G. If in the (internal) direct decomposition

$$G = \mathrm{GL}(l_1, p^l)^{\sigma_1} \times \dots \times \mathrm{GL}(l_r, p^l)^{\sigma_r},$$

$\mathrm{GL}(l_1, p^l)$ is replaced by $\langle \eta_1 \rangle \times U$, the factor $\langle {\eta_1}^{\sigma_1} \rangle$ in the resulting decomposition can evidently be replaced by $\langle \hat{\eta} \rangle$, since $\hat{\eta} \in Z(G)$, $\hat{\eta}$ has order q and $\hat{\eta}^{\pi_1} = \eta_1$. Hence $\langle \hat{\eta} \rangle$ is a direct factor of G, and therefore $\langle \tilde{\eta} \rangle$ is a direct factor of $C_{\mathrm{GL}(n,p^l)}(\tilde{\eta})$. Hence, passing to conjugates, we see that $\langle \eta \rangle$ is a direct factor of $C_{\mathrm{GL}(n,p^l)}(\eta)$. *A fortiori*, $\langle \eta \rangle$ is a direct factor of a Sylow q-subgroup of $C_{\mathrm{GL}(n,p)}(\eta)$ and therefore by Theorem 3.5 all extensions of E split. This establishes (i).

In order to prove (ii), it is enough by the Note after Theorem 3.5 to show that under the conditions stated η is a qth power in $\mathrm{GL}(n, p)$. Thus the proof of Theorem 3.10 is completed by establishing

LEMMA 3.13. *With the notation of Theorem* 3.10, *suppose that either* $p^l \equiv 1\ (q^2)$ *or* q *divides* l_i *for all* $i = 1, \dots, r$. *Then* η *is a* q*th power in* $\mathrm{GL}(n, p)$.

Proof. It is convenient to change from the multiplicative notation for A to additive notation: so let V be a vector space of dimension n over

GF(p). Regard η as an automorphism of V (by referring to an arbitrarily chosen base of V). Then, since $p \neq q$, by Maschke's theorem the $\langle\eta\rangle$-module V is completely reducible: say

$$V = V_1 \oplus \dots \oplus V_m,$$

where $V_1, \dots, V_m$ are irreducible $\langle\eta\rangle$-submodules of V. It is easy to verify (and must be well known) that every irreducible representation of a cyclic group of order q over GF(p) is equivalent to either the 1-representation or one of $(q-1)/t$ representations of degree t defined by mapping a generator of the group to the multiplications of GF(p^t) by $(q-1)/t$ suitable elements of GF(p^t). Since η is fixed-point-free, no V_j can afford the 1-representation. Therefore

$$\dim V_j = t \quad \text{for } j = 1, \dots, m$$

and

$$n = mt.$$

Let η_j be the restriction of η to V_j, for $j = 1, \dots, m$. Then each η_j has order q and belongs (by natural identification) to GL(t, p). However, the Sylow q-subgroups of GL(t, p) are cyclic of order q^s, where q^s is the highest power of q dividing $p^t - 1$. Therefore, if $s > 1$, η_j is certainly a qth power in GL(t, p), that is there is an automorphism ξ_j of V_j such that

$$\xi_j{}^q = \eta_j \quad \text{for } j = 1, \dots, m.$$

Then an automorphism ξ of V can be defined by prescribing that its restriction to V_j coincide with ξ_j for all j. It is clear that then

$$\xi \in \mathrm{GL}(n, p) \quad \text{and} \quad \xi^q = \eta.$$

Thus it may be assumed that $s = 1$, in which case by hypothesis every l_i is a multiple of q. Every V_j affords a non-trivial irreducible representation of $\langle\eta\rangle$ over GF(p) and hence determines an element μ_j of GF(p^t), where the action of η on V_j is equivalent to multiplication of GF(p^t) by μ_j. Then μ_j must be an eigenvalue of η, occurring corresponding to V_j with multiplicity t. Therefore each l_i is a multiple of t. Because q and t are relatively prime, it follows that each l_i is a multiple of qt. Hence the number of V_j for which the corresponding μ_j are all equal to any particular eigenvalue λ_i is a multiple of q. In particular, q divides m, say $m = qk$.

Now V can be decomposed as

$$V = W_1 \oplus \dots \oplus W_k,$$

where each $W_j = V_{j_1} \oplus \dots \oplus V_{j_q}$ for suitable $j_1, \dots, j_q$ such that $\mu_{j_1} = \dots = \mu_{j_q}$, that is such that $V_{j_1}, \dots, V_{j_q}$ are isomorphic $\langle\eta\rangle$-modules. It is obviously sufficient to complete the proof to show that, for each $j = 1, \dots, k$, the restriction of η to W_j is a qth power in GL(qt, p). Therefore it may be assumed for the purpose of the argument that $k = 1$, so that

$$V = V_1 \oplus \ldots \oplus V_q,$$

where $V_1, \ldots, V_q$ are isomorphic irreducible $\langle\eta\rangle$-submodules of V.

Now there is an automorphism ζ of V of order q which permutes the submodules $V_1, \ldots, V_q$ cyclically and has no other effect. Further let $\eta_1{}^*$ be the automorphism of V whose restriction to V_1 coincides with η_1 and whose restriction to V_j for $j \geqslant 2$ is the identity map on V_j. Then it is easy to verify that

$$(\eta_1{}^*\zeta)^q = \eta.$$

(This is effectively just a calculation in the regular wreath product $C_q \wr C_q$, where C_q denotes a group of order q. The Sylow q-subgroups of $\mathrm{GL}(q, p)$ are isomorphic to $C_q \wr C_q$ when $s = 1$ as above: see Weir [**13**].) This completes the proof of Lemma 3.13 and thus also of Theorem 3.10.

Theorem 3.10 can be applied to give in certain special circumstances a criterion for a finite group G to split over the second term from the bottom of a chief series of G. The following lemma will be useful.

LEMMA 3.14. *Let L be a finite group with an abelian normal subgroup N of prime index q in L, where q does not divide $|N|$. Let y be an element of L of order q. Then $C_N(y)$ is a characteristic subgroup of L.*

Proof. Let $\alpha \in \operatorname{Aut} L$. Since $\langle y\rangle$ is a Sylow q-subgroup of L, $\langle y^\alpha\rangle$ is also a Sylow q-subgroup of L and therefore, because $L = \langle y\rangle N$,

$$\langle y^\alpha\rangle = \langle y^x\rangle \quad \text{for some } x \in N.$$

Certainly N is a characteristic subgroup of L, and so

$$(C_N(y))^\alpha = C_N(y^\alpha) = C_N(y^x) = (C_N(y))^x = C_N(y),$$

since N is abelian.

THEOREM 3.15. *Let G be a finite group with an abelian minimal normal subgroup N, say of order p^n, and a chief factor L/N of prime order $q \neq p$, such that L is non-abelian. Let t be the order of p (mod q). If $p^t \not\equiv 1$ (q^2) and q does not divide n, then G splits over L.*

Proof. Let y be an element of L of order q. By Lemma 3.14, $C_N(y)$ is a normal subgroup of G, and hence by the minimality of N either $C_N(y) = 1$ or $C_N(y) = N$. The latter alternative would contradict the hypothesis that L is non-abelian. Hence $C_N(y) = 1$, which means that the action of y on N by conjugation is fixed-point-free. Therefore L is isomorphic to the group E of Theorem 3.10 when $A = N$ and η is conjugation in N by y. If the multiplicities of the distinct eigenvalues of η in $\mathrm{GF}(p^t)$ are $l_1, \ldots, l_r$

respectively, then since $\sum_{i=1}^{r} l_i = n$ and by hypothesis q does not divide n, some l_i is not divisible by q. Then it follows from Theorem 3.10 that G splits over L.

An exactly similar argument, using Lemma 3.14 in conjunction with the fact noted after Lemma 3.2 that every extension of E by every finite q'-group splits (without any condition on E other than its definition) shows that *if G is a finite group for which the prime q is a divisor of $|G|$ but q^2 is not a divisor of $|G|$, and if G has an abelian minimal normal subgroup N and a chief factor L/N of order q such that L is non-abelian, then G splits over L.*

But in general, the arithmetical conditions in Theorem 3.15 are relevant. To see this, consider an elementary abelian group A of order p^n, where p is an odd prime and n any positive integer, and let $G = \operatorname{Hol} A$. Let q be any prime divisor of $p-1$. Then A is a minimal normal subgroup of G, G/A has a unique normal subgroup L/A of order q and L is non-abelian. In fact $L/A \leqslant Z(G/A)$, which is cyclic of order $p-1$, so that if $p \equiv 1\ (q^2)$ then $Z(G/A)$ has a unique cyclic subgroup K/A of order q^2; then G cannot split over L because $K/A > L/A$ but K/A does not split over L/A.

Now suppose that $p \not\equiv 1\ (q^2)$ but that n is a multiple of q. With η as the unique element of order q in $Z(\operatorname{Aut} A)$,

$$L = \langle \eta \rangle A = E \quad \text{in Theorem 3.10.}$$

In the notation of that theorem, $t = 1$, $r = 1$, and $l_1 = n$. Then, since q divides l_1, Lemma 3.13 shows that

$$\eta = \xi^q \quad \text{for some } \xi \in \operatorname{Aut} A.$$

Let $K = \langle \xi \rangle A < G$. Then again G cannot split over L because $K/A > L/A$ but K/A does not split over L/A.

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School of Mathematics
University of Newcastle upon Tyne

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3.

EXTENSIONS BY A FREE ABELIAN GROUP OF RANK 2

By J. S. Rose

Department of Pure Mathematics, The University, Newcastle-upon-Tyne

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Abstract

It has been shown that for any group K with non-trivial centre, there is a non-split extension of K by some abelian group A. There is evidence for expecting that A may be chosen to be free abelian of rank 2. A general criterion for all extensions of a group by a free abelian group of rank 2 to split is established. Then it is shown that a group with non-trivial centre may in fact have all its extensions by a free abelian group of rank 2 split.

1°. A group G is called *complete* if it has trivial centre and if all its automorphisms are inner. It was shown by O. Hölder [2, p. 94] that if G is complete then G is a direct factor of every group containing it as a normal subgroup. R. Baer [1] proved conversely that if G is a direct factor of every group containing it as a normal subgroup then G must be complete. Indeed, in this converse direction, Baer showed by appeal to a simple result of extension theory [6, p. 131] that for any group K with non-trivial centre there is a group H containing K as a normal subgroup but not a direct factor and such that H/K is free abelian of rank 2.

More recently in [5] it has been shown that every group containing a group G as a normal subgroup splits over G (as a semi-direct, not necessarily direct product) if and only if G has trivial centre and the full automorphism group of G splits over the group of inner automorphisms of G. In this connexion it has also been shown [5, Corollary 2.6] that if K is a group with non-trivial centre then there is a group H containing K as a normal subgroup such that H/K is abelian and H does not split over K. The question is posed: can we even arrange that H/K is free abelian of rank 2, as in Baer's argument?

Throughout this paper let F denote a free abelian group of rank 2.

2°. As circumstantial evidence we might adduce the fact (pointed out to the author by Professor K. W. Gruenberg) that for F acting trivially on an arbitrary abelian group A, the corresponding 2-cohomology group is isomorphic to A:

$$H^2(F, A) \cong A.$$

(This can be seen by a direct application of the universal coefficient theorem for cohomology, together with a theorem of H. Hopf: see [3, Theorem and formula (5)]). Now, according to the theory of S. Eilenberg and S. MacLane (see for example [4, §§ 49 and 51]), the equivalence classes of extensions of a group K by a group Q with a fixed coupling θ of Q to K (in the terminology of [5]) are, if they exist at all, in one-to-one correspondence with the elements of $H^2(Q, Z(K))$, where $Z(K)$ denotes the centre of K and the action of Q on $Z(K)$ is induced naturally by θ; and hence with the equivalence classes of extensions of $Z(K)$ by Q with coupling θ_1 induced naturally by θ. If θ is trivial then there are corresponding extensions of K by Q, for instance the direct product; and of course θ_1 is trivial. Hence, if $Z(K) \neq 1$ and $Q = F$, there must be extensions of K by F with trivial coupling which correspond to a non-trivial element of $H^2(F, Z(K))$ and hence to non-split extensions of $Z(K)$ by F. Then we might expect that one of these extensions of K by F with trivial coupling would be non-split. But there is a difficulty: for it is not clear that split extensions of K by F necessarily correspond to split extensions of $Z(K)$ by F. In fact we shall show that a group K with non-trivial centre may have all its extensions by F split (for all possible couplings of F to K). This will answer negatively the question posed in 1°.

3°. In what follows, we make no use of the notion of equivalence of extensions. It is therefore sufficient for our purposes to take an *extension* of a group A by a group B to mean a group E containing A as a normal subgroup and such that $E/A \cong B$. Then E is a *split* extension of A by B if E *splits* over A, that is if there is a subgroup C of E such that $E=AC$ and $A \cap C=1$. To say that an extension E of A by B has *trivial coupling* means simply that $E=A\,C_E(A)$, where $C_E(A)$ denotes the centralizer of A in E.

We shall use the following standard notation. If x and y are elements of a group G, then

$$x^y=y^{-1}xy \text{ and } [x, y]=x^{-1}x^y.$$

If $x_1, x_2, \ldots\ldots, x_n$ are elements of G, then $\langle x_1, x_2, \ldots\ldots, x_n\rangle$ denotes the subgroup of G generated by these elements.

For each element u of a fixed group K, we shall denote by v_u the inner automorphism of K induced by u.

PROPOSITION 1. *Let K be a group. All extensions of K by F split if and only if for every pair σ, τ of automorphisms of K and each element u of K such that $[\sigma, \tau]=v_u$ there exist elements a, b of K such that*

$$a^\sigma ba^{-1}\,b^{-\tau}=u.$$

Proof. Suppose first that G is an extension of K by F, and let H/K be an infinite cyclic direct factor of G/K. Then H splits over K, say

$$H=\langle s\rangle\,K \text{ with } s \in H \text{ and } \langle s\rangle \cap K = 1,$$

and since G/H is infinite cyclic, G splits over H, say

$$G=\langle t\rangle\,H \text{ with } t \in G \text{ and } \langle t\rangle \cap H=1.$$

Then

$$G/K=\langle sK, tK\rangle,$$

and since this is abelian,

$$u=[s, t]\,\varepsilon\, K.$$

Let σ, τ be the automorphisms of K induced respectively by conjugation in G by s, t. Then

$$[\sigma, \tau]=\nu_u.$$

Now suppose that there are elements a, b of K such that

$$a^\sigma ba^{-1}b^{-\tau}=u. \tag{1}$$

Define elements s_1, t_1 of G by

$$s_1=sb \text{ and } t_1=ta.$$

Then

$$s_1t_1=sbta=ts^tb^ta=tsub^\tau a, \tag{2}$$

by definition of u and τ, and

$$t_1s_1=tasb=tsa^sb=tsa^\sigma b, \tag{3}$$

by definition of σ. Thence by (1),

$$s_1t_1=t_1s_1,$$

so that the subgroup $J=\langle s_1, t_1\rangle$ of G is abelian. Therefore every element of J has the form $s_1{}^nt_1{}^m$ for suitable integers n, m. If $s_1{}^n\, t_1{}^m\,\varepsilon\, K$ then $t_1{}^m\,\varepsilon\, H$ and this implies that $m=0$ because for any positive integer r, $t_1{}^r=t^ra_r$ with $a_r\,\varepsilon\, K$; then $s_1{}^n\,\varepsilon\, K$, and this implies similarly that $n=0$. Hence

$$J\cap K=1.$$

But also

$$J\,K=\langle s, t\rangle\, K=G,$$

and so G splits over K. This proves the sufficiency of the condition for all extensions of K by F to split.

In order to prove the necessity, suppose that σ, τ are automorphisms of K and $u\,\varepsilon\, K$ with $[\sigma, \tau]=\nu_u$ but such that there are no elements a, b of K with

$$a^\sigma ba^{-1}b^{-\tau}=u.$$

Let $\langle s\rangle$ be an infinite cyclic group, and let H be the semi-direct product of K by $\langle s\rangle$ with action of $\langle s\rangle$ on K defined by the condition that s induces σ: that is, in $H=\langle s\rangle\, K$,

$$a^s=a^\sigma \text{ for all elements } a \text{ of } K.$$

Now we extend τ to a map $\hat{\tau}$ of H into itself, defined by

$$\hat{\tau}: s^n\, a \mapsto (su)^n\, a^\tau$$

for all elements a of K and all integers n. The map $\hat{\tau}$ is an endomorphism of H. To show this, let $a, b\,\varepsilon\, K$ and let n, m be integers: then

$$(s^na)\,(s^mb)=s^{n+m}a^{\sigma^m}b \overset{\hat{\tau}}{\mapsto} (su)^{n+m}(a^{\sigma^m}b)^\tau=(su)^n(su)^m a^{\sigma^m\tau}b^\tau,$$

and we must verify that

$$(su)^m a^{\sigma^m\tau}=a^\tau(su)^m.$$

By hypothesis, $[\sigma, \tau]=v_u$, so that

$$\tau^{-1}\sigma\tau=\sigma v_u$$

and hence for every integer r,

$$\tau^{-1}\sigma^r\tau=(\sigma v_u)^r.$$

Since

$$a^{\sigma v_u} = (su)^{-1}a(su),$$

$$a^{(\sigma v_u)^r}=(su)^{-r}a(su)^r.$$

Therefore

$$(su)^m a^{\sigma^{m}\tau}=(su)^m a^{\tau(\sigma v_u)^m}=a^\tau(su)^m$$

as required. It is clear that $\hat{\tau}$ is bijective, so that in fact $\hat{\tau}$ is an automorphism of H.

Let $\langle t\rangle$ be an infinite cyclic group, and let G be the semi-direct product of H by $\langle t\rangle$ with action of $\langle t\rangle$ on H defined by the condition that t induces $\hat{\tau}$. Since $\hat{\tau}$ maps K into itself, K is a normal subgroup of G. Moreover,

$$G/K=\langle s\,K,\ t\,K\rangle$$

and this quotient group is abelian since in G,

$$[s, t]=s^{-1}\,s^{\hat{\tau}}=u\in K.$$

In G, no positive power of either s or t belongs to K, and $\langle sK\rangle\cap\langle tK\rangle=K/K$; hence $G/K\cong F$.

Finally, G does not split over K: for if it did, there would be a subgroup J of G such that $G=J\,K$ and $J\cong F$. Then there would be elements s_1, t_1 in J and a, b in K such that

$$s_1=sb \text{ and } t_1=ta.$$

As in (2) and (3), we should find that

$$s_1t_1=tsub^\tau a \text{ and } t_1s_1=tsa^\sigma b.$$

But J would be abelian and so

$$s_1t_1=t_1s_1,$$

whence

$$u=a^\sigma ba^{-1}b^{-\tau},$$

contrary to hypothesis.

A similar argument yields

PROPOSITION 2. *Let K be a group. All extensions of K by F with trivial coupling split if and only if every element of $Z(K)$ is a commutator in K.*

Proof. Suppose that G is an extension of K by F with trivial coupling, and let H/K be an infinite cyclic direct factor of G/K. Then H splits over K, say

$$H=\langle h\rangle K \text{ with } h\in H \text{ and } \langle h\rangle\cap K=1.$$

By the hypothesis of trivial coupling, $h=ks$ for some $k\in K$ and $s\in C_G(K)$. Then $H=\langle s\rangle\,K$ and $\langle s\rangle\cap K=1$, so that $H=\langle s\rangle\times K$, the direct product. Since G/H is infinite cyclic, G splits over H, say

$$G=\langle t\rangle\,H \text{ with } t\in G \text{ and } \langle t\rangle\cap H=1.$$

Now G/K is abelian and $s \varepsilon C_G(K)$ so that

$$u=[s, t] \varepsilon K \cap C_G(K)=Z(K).$$

Since $G=KC_G(K)$ the automorphism τ of K induced by conjugation in G by t is an inner automorphism of K, say

$$\tau=v_w \text{ with } w \varepsilon K.$$

If all elements of $Z(K)$ are commutators in K, then $u=[x, y]$ for some $x, y \varepsilon K$. Hence

$$u=u^w=(xw)^{-1}y^{-1}(xw)y^w,$$

that is

$$u=a^\sigma ba^{-1}b^{-\tau}, \tag{1}$$

where $a=w^{-1}x^{-1} \varepsilon K$, $b=y^{-1} \varepsilon K$ and $\sigma=1$, the automorphism of K induced by conjugation in G by s. This equation (1) is the same as the equation (1) in the proof of Proposition 1, and we can now show that G splits over K exactly as in the previous proof.

Now suppose that there is an element z of $Z(K)$ which is not a commutator in K. Let $\langle s\rangle$ be an infinite cyclic group and let H be the direct product

$$\langle s\rangle \times K.$$

It is clear that the map

$$\alpha: s^n a \mapsto s^n z^{-n} a$$

(defined for all elements a of K and all integers n) is an automorphism of H. Let $\langle t\rangle$ be an infinite cyclic group, and let G be the semi-direct product of H by $\langle t\rangle$ with action of $\langle t\rangle$ on H such that t induces α. Then K is a normal subgroup of G, and as in the proof of Proposition 1 (with α in place of $\hat{\tau}$) $G/K \cong F$ and G does not split over K: for it if did we should have $z^{-1}=aba^{-1}b^{-1}$ with $a, b \varepsilon K$, and then $z = [b^{-1}, a^{-1}]$, a commutator in K. Finally, the coupling of the extension constructed is trivial, for s and t centralize K, hence

$$G=K\,C_G(K).$$

4°. Let Q denote the quaternion group (of order 8). It follows at once from Proposition 2 that every extension of Q by F with trivial coupling splits. On the other hand, we can show that there are extensions of Q by F which do not split. With the usual notation for Q, let i, j, k denote generators of the three subgroups of Q of order 4. Q has an automorphism σ of order 3 which permutes the elements i, j, k cyclically. Let τ be the identity automorphism of Q, and let $u=-1$, the unique element of Q of order 2. In order to show that there is a non-split extension of Q by F it will be enough, according to Proposition 1, to show that there are no elements a, b of Q such that

$$a^\sigma ba^{-1}b^{-\tau}=u,$$

that is, such that

$$a^\sigma=-(bab^{-1}).$$

Now σ fixes the elements 1 and -1 of Q and since they are 'self-conjugate' elements of Q, a could not be either of these. But if a were any of the other six elements of Q, the conjugates of a in Q would be just a and $-a$ whereas σ would move a outside the subgroup $\langle a\rangle=\{1, a, -1, -a\}$. Hence there are indeed no such elements a, b.

5°. We shall show finally that it can even happen that all extensions by F of a group with non-trivial centre split.

PROPOSITION 3. *Let D denote the dihedral group of order* 8. *All extensions of D by F split.*

In order to facilitate the application of Proposition 1, we introduce the following terminology for a given group K: an ordered triple (σ, τ, u) consisting of automorphisms σ and τ of K and an element u of K such that $[\sigma, \tau]=v_u$ will be called a *triple for* K; and it will be called a *splitting triple* if in addition there are elements a and b of K such that $a^\sigma ba^{-1}b^{-\tau}=u$. Then the assertion of Proposition 1 is that all extensions of K by F split if and only if every triple for K is a splitting triple.

Now we note some simple facts. Let (σ, τ, u) always denote a triple for K.

(i) *(σ, τ, u) is a splitting triple if there is an element a of K such that $a^\sigma a^{-1}=u$.*

(ii) *(σ, τ, u) is a splitting triple if there is an element b of K such that $bb^{-\tau}=u$.*

(iii) *If (σ, τ, u) is a splitting triple then (τ, σ, u^{-1}) is also a splitting triple.*

We observe that (τ, σ, u^{-1}) is a triple for K, since

$$[\tau, \sigma]=[\sigma, \tau]^{-1}={v_u}^{-1}=v_{u^{-1}}$$

and then that if a, b are elements of K such that $a^\sigma ba^{-1}b^{-\tau}=u$; b, a are elements of K such that $b^\tau ab^{-1}a^{-\sigma}=u^{-1}$.

(iv) *If (σ, τ, u) is a splitting triple and if c is an element of K such that $c^\tau=c$ then $(v_c\sigma, \tau, u)$ is a splitting triple.*

To verify this, note first that for any x in K,

$$x^{v_c\tau}=(c^{-1}xc)^\tau=c^{-1}x^\tau c=x^{\tau v_c}$$

so that

$$[v_c, \tau]=1.$$

Hence $[v_c\sigma, \tau]=[v_c, \tau]^\sigma[\sigma, \tau]=[\sigma, \tau]=v_u$, and so $(v_c\sigma, \tau, u)$ is a triple for K. Now if a, b are elements of K such that $a^\sigma ba^{-1}b^{-\tau}=u$ then

$$(cac^{-1})^{v_c\sigma}\,(bc^{-1})\,(ca^{-1}c^{-1})\,cb^{-\tau}=u,$$

that is

$$(cac^{-1})^{v_c\sigma}\,(bc^{-1})\,(cac^{-1})^{-1}\,(bc^{-1})^{-\tau}=u.$$

Thus $(v_c\sigma, \tau, u)$ is a splitting triple.

A similar argument gives

(v) *If (σ, τ, u) is a splitting triple and if d is an element of K such that $d^\sigma=d$ then $(\sigma, v_d\tau, u)$ is a splitting triple.*

Proof of Proposition 3. We must show that every triple (σ, τ, u) for D is a splitting triple.

We can write

$$D=\langle x, t\rangle \text{ where } x^4=t^2=1 \text{ and } x^t=x^{-1}.$$

It is easy to show that the full automorphism group A of D is isomorphic to D: in fact

$$A=\langle \alpha, v_t\rangle,$$

where α is the automorphism of D defined by

$$x^\alpha=x \text{ and } t^\alpha=xt,$$

and v_t is the inner automorphism of D induced by t. Then $\alpha^4=v_t{}^2=1$ and $\alpha^{v_t}=\alpha^{-1}$. Note that $\alpha^2=v_x$.

The only commutators in A are 1 and v_x. Then since $Z(D)=\langle x^2\rangle$ the triples for D are just the triples (σ, τ, u), where $\sigma, \tau \in A$ and either $[\sigma, \tau]=1$ and $u=1$ or x^2, or $[\sigma, \tau]=v_x$ and $u=x$ or x^{-1}. Let (σ, τ, u) be a triple for D.

If $u=1$ then (σ, τ, u) is trivially a splitting triple.

Suppose that $u=x$. Then $\sigma, \tau \notin Z(A)=\langle \alpha^2\rangle$. Note that

$$t^\alpha t^{-1}=x,$$

$$(xt)^{\alpha v_t}(xt)^{-1}=tx^2tt(xt)=x,$$

and

$$t^{\alpha^{-1}v_t}t^{-1}=tx^{-1}ttt=x.$$

Hence by (i), a triple for D of the form (σ, τ, x) is a splitting triple if $\sigma=\alpha$ or αv_t or $\alpha^{-1}v_t$. Also

$$tt^{-\alpha^{-1}}=tx^{-1}t=x,$$

$$tt^{-\alpha v_t}=t(txtt)=x,$$

and

$$(xt)\,(xt)^{-\alpha^{-1}v_t}=(xt)t=x.$$

Hence by (ii), a triple for D of the form (σ,τ, x) is a splitting triple if $\tau=\alpha^{-1}$ or αv_t or $\alpha^{-1}v_t$. The only remaining triples to consider with $u=x$ are (α^{-1}, v_t, x), (α^{-1}, v_{xt}, x), (v_t, α, x) and (v_{xt}, α, x). Since

$$(xt)^{\alpha^{-1}}t(xt)^{-1}\,t^{-v_t}=tt(xt)t=x,$$

and

$$t^{v_t}(xt)\,t^{-1}(xt)^{-\alpha}=t(xt)\,t(x^2t)=x,$$

(α^{-1}, v_t, x) and (v_t, α, x) are both splitting triples. Then since $x^{\alpha^{-1}}=x$ and $x^\alpha=x$, (v) shows that (α^{-1}, v_{xt}, x) is a splitting triple and (iv) that (v_{xt}, α, x) is a splitting triple.

Now suppose that $u=x^{-1}$. If (σ, τ, x^{-1}) is a triple for D, then so is (τ, σ, x). Then by the previous paragraph (τ, σ, x) is a splitting triple and therefore by (iii), (σ, τ, x^{-1}) is a splitting triple.

Suppose finally that $u=x^2$. We note first that

$$t^{\alpha^2}t^{-1}=x^2,$$

$$x^{v_t}x^{-1}=x^{-2}=x^2,$$

$$x^{\alpha v_t}x^{-1}=x^{v_t}x^{-1}=x^2,$$

$$x^{v_{xt}}\,x^{-1}=x^{v_t}x^{-1}=x^2,$$

and

$$x^{\alpha^{-1}v_t}\,x^{-1}=x^{v_t}x^{-1}=x^2.$$

Hence by (i), all triples for D of the form (σ, τ, x^2) with $\sigma=\alpha^2$ or v_t or αv_t or v_{xt} or $\alpha^{-1}v_t$ are splitting triples. Next, since $x^{-2}=x^2$, (iii) shows that all triples for D of the form (σ, τ, x^2) with $\tau=\alpha^2$ or v_t or αv_t or v_{xt} or $\alpha^{-1}v_t$ are splitting triples.

The only remaining triples to consider are those with $u=x^2$ and $\sigma, \tau \in \{1, \alpha, \alpha^{-1}\}$. Any such τ fixes x and so

$$txt^{-1}x^{-\tau}=x^{-1}\,x^{-1}=x^2.$$

Hence $(1, 1, x^2)$, $(1, \alpha, x^2)$ and $(1, \alpha^{-1}, x^2)$ are splitting triples, and then by (iii), $(\alpha, 1, x^2)$ and $(\alpha^{-1}, 1, x^2)$ are also splitting triples. Next,

$$t^{\alpha}(xt)\,t^{-1}(xt)^{-\alpha}=(xt)\,(xt)\,t(x^2t)=x^2,$$

so that (α, α, x^2) is a splitting triple. Then since $\alpha^{-1}=v_x\alpha$ and $x^{\alpha}=x$, (iv) shows that $(\alpha^{-1}, \alpha, x^2)$ is a splitting triple, then (iii) that $(\alpha, \alpha^{-1}, x^2)$ is a splitting triple, and finally (iv) again that $(\alpha^{-1}, \alpha^{-1}, x^2)$ is a splitting triple. This covers all cases and completes the proof.

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A SUBNORMAL EMBEDDING THEOREM FOR FINITE GROUPS

JOHN S. ROSE

Let G be a group and $\mathfrak{X}$ a class of groups. One says that G can be *embedded subnormally* in an $\mathfrak{X}$-group if there is a group X in the class $\mathfrak{X}$ which contains G as a subnormal subgroup. R. S. Dark [**3**] has shown, for instance, that any group can be embedded subnormally in a perfect group, but that the symmetric group of degree 3 cannot be embedded subnormally in a finite perfect group.

This note is concerned with the possibility of embedding finite groups subnormally in complete finite groups. A group G is *complete* if it has trivial centre and if every automorphism of G is inner. There is a classical theory of complete groups due to O. Hölder: see W. Burnside [**2**; §§70, 71, 72, 1962]. In this connexion there is a subnormal embedding result which is an immediate consequence of the well-known automorphism tower theorem of H. Wielandt [**7**; (45)]. Since the terminal member of the automorphism tower of a finite group with trivial centre is evidently a complete group, any finite group with trivial centre can be embedded subnormally in a complete finite group.

It will be shown here that any finite group can be embedded subnormally in a complete finite group. In fact, the following more specific result will be established. *Let ϖ be any set of at least two prime numbers and let G be any finite ϖ-group. Then there is a complete finite group K with a normal Hall ϖ-subgroup J containing G as a subnormal subgroup. Moreover, K/J is soluble; and if G is soluble then K can be made soluble.*

I do not know whether K can always be made a ϖ-group. Indeed, I do not even know whether complete groups of odd orders exist.

Let G be any group. The group of all automorphisms of G is denoted by $\operatorname{Aut} G$, the group of all inner automorphisms of G by $\operatorname{Inn} G$, and $\operatorname{Out} G = \operatorname{Aut} G/\operatorname{Inn} G$. If $H \lhd G$ and $C_G(H) = 1$ then there is a natural isomorphism of G onto a subgroup of $\operatorname{Aut} H$, by which H is mapped to $\operatorname{Inn} H$. This isomorphism will be used to identify G with the appropriate subgroup of $\operatorname{Aut} H$ and H with $\operatorname{Inn} H$: then $H \leqslant G \leqslant \operatorname{Aut} H$. In particular, when $Z(G) = 1$, $G \lhd \operatorname{Aut} G$.

If G is finite then $G/R(G)$ will denote the *soluble residual* of G: that is, $R(G)$ is the smallest normal subgroup of G for which $G/R(G)$ is soluble. It is a simple observation that if H is a subnormal subgroup of G and if there is a chain of subgroups

$$H = H_0 \leqslant H_1 \leqslant H_2 \leqslant \ldots \leqslant H_n = G$$

such that $H_{j-1} \lhd H_j$ and H_j/H_{j-1} is soluble for each $j = 1, \ldots, n$ then $R(G) \leqslant H$.

The result stated above follows at once from the

THEOREM. *Let ϖ be any set of at least two prime numbers and let G be any finite ϖ-group. Then G can be embedded subnormally in a finite ϖ-group J such that $Z(J) = 1$ and* Out *J is a ϖ'-group. Moreover, there is a complete group K such that*

$$J \leqslant K \leqslant \operatorname{Aut} J,$$

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and there is a subnormal subgroup E of J such that $R(K) \leqslant E$ and E is a central product of copies of G.

The last clause of this statement means that E is the join of normal subgroups each isomorphic to G and which centralize each other.

The first step in the proof is to show that any finite ϖ-group can be embedded subnormally in a finite ϖ-group with trivial centre. The method is by a wreath product construction. My argument has been simplified in the light of a helpful remark of Dr. Brian Hartley.

Let T be any group permuting a set X, say by means of the permutation representation ρ. Let G be any group $\neq 1$, and let $D = \mathrm{Dr}\, G^X$, the (restricted) direct power. There is a corresponding action of T on D: let W be the semi-direct product of D by T with this action. (See, for instance, **[5]**.) Then W is the (restricted) wreath product of G by T determined by ρ, and D is the base group of W. Each element of W is uniquely expressible in the form tf with $t \in T$ and $f \in D$. Straightforward calculations show that $tf \in Z(W)$ if and only if $t \in Z(T) \cap \mathrm{Ker}\, \rho$, $f(x) \in Z(G)$ for all $x \in X$ and f is constant on each T-orbit of X.

Now suppose that T permutes X faithfully and transitively and that $|X|$ is finite and $\neq 1$. Then

$$\begin{aligned} Z(W) &= \{f \in D \mid f \text{ is a constant function and the value of } f \text{ is in } Z(G)\} \\ &\leqslant Z(D). \end{aligned}$$

Since $|X| > 1$, $D/Z(W)$ is certainly a central product of copies of G. Therefore G can be embedded subnormally in $W/Z(W)$.

Let

$$Z(W/Z(W)) = Z_2(W)/Z(W).$$

Lemma 1 characterizes $Z_2(W)$.

LEMMA 1. *Suppose that the group T permutes the finite set X faithfully and transitively, that $|X| \neq 1$ and that G is a group $\neq 1$. Let W be the corresponding wreath product of G by T, and let $D = \mathrm{Dr}\, G^X$, the base group of W. Then if $|X| > 2$ or if $|X| = 2$ and G is non-abelian, $Z_2(W) = \{f \in D \mid f(x) \in Z(G)$ for all $x \in X$ and there is a homomorphism $\zeta : T \to Z(G)$ such that $f(xt) = f(x)\, t^{\zeta}$ for all $x \in X$ and all $t \in T\}$.*

Proof. Let $t \in T$ and $f \in D$. Then $tf \in Z_2(W)$ if and only if $[t_1, tf] \in Z(W)$ and $[tf, f_1] \in Z(W)$ for all $t_1 \in T$ and all $f_1 \in D$. Now

$$[t_1, tf] = t_1^{-1} f^{-1} t^{-1} t_1\, tf = [t_1, t][t^{-1} t_1\, t, f].$$

Since $[t_1, t] \in T$ and $[t^{-1} t_1\, t, f] \in D$, it follows that $[t_1, tf] \in Z(W)$ for all $t_1 \in T$ if and only if $t \in Z(T)$ and $[t_1, f] \in Z(W)$ for all $t_1 \in T$.

If the latter condition is satisfied then for each $t_1 \in T$ and for all $x \in X$, $f(xt_1^{-1})^{-1} f(x)$ is independent of x and its constant value is in $Z(G)$, say

$$f(xt_1^{-1})^{-1} f(x) = z_{t_1} \in Z(G).$$

Then

$$f(xt_1) = f(x) z_{t_1} \quad \text{for all } x \in X \text{ and all } t_1 \in T.$$

Let $t_1, t_2 \in T$ and $x \in X$: then

$$f(x)z_{t_1 t_2} = f(xt_1 t_2) = f(xt_1)z_{t_2} = f(x)z_{t_1}z_{t_2}.$$

Hence the map $\zeta : t_1 \mapsto z_{t_1}$ is a homomorphism of T into $Z(G)$. Conversely, if there is a homomorphism $\zeta : T \to Z(G)$ such that

$$f(xt_1) = f(x)\, t_1{}^{\zeta} \text{ for all } x \in X \text{ and all } t_1 \in T,$$

then clearly $[t_1, f] \in Z(W)$ for all $t_1 \in T$.

Now suppose that $[tf, f_1] \in Z(W)$ for all $f_1 \in D$. Since $[tf, f_1] = f^{-1}t^{-1}f_1{}^{-1}tff_1$, this means that for each $f_1 \in D$ and for all $x \in X$, $f(x)^{-1}f_1(xt^{-1})^{-1}f(x)f_1(x)$ is independent of x and its constant value is in $Z(G)$.

Assume first that G is non-abelian. Then if $xt^{-1} \neq x$ for some $x \in X$, there would be a function $f_1 \in D$ such that $f_1(xt^{-1}) = 1$ and $f_1(x) \notin Z(G)$: but then the condition $f(x)^{-1}f_1(xt^{-1})^{-1}f(x)f_1(x) \in Z(G)$ would be violated. Hence $xt^{-1} = x$ for all $x \in X$, and therefore, since T permutes X faithfully, $t = 1$. By hypothesis $|X| \geqslant 2$: let $x_1, x_2 \in X$ with $x_1 \neq x_2$, and let $g \in G$. There is a function $f_1 \in D$ such that $f_1(x_1) = 1$ and $f_1(x_2) = g$: since $t = 1$ the condition above gives

$$[f(x_2), g] = [f(x_1), 1] = 1.$$

Since x_2 and g can be chosen arbitrarily, it follows that $f(x) \in Z(G)$ for all $x \in X$.

Now assume that G is abelian: then by hypothesis $|X| \geqslant 3$. The condition above becomes that for each $f_1 \in D$ and for all $x \in X$, $f_1(xt^{-1})^{-1}f_1(x)$ is independent of x. If t fixes some point $x_0 \in X$, then, for all $f_1 \in D$ and all $x \in X$,

$$f_1(xt^{-1})^{-1}f_1(x) = f_1(x_0 t^{-1})^{-1}f_1(x_0) = 1;$$

hence

$$xt^{-1} = x \text{ for all } x \in X,$$

and so, since T permutes X faithfully, $t = 1$. Suppose that $t \neq 1$: then t fixes no point of X. If there were a point $x_1 \in X$ such that $x_1 t^{-1} \neq x_1 t$, then, since also $x_1 t^{-1} \neq x_1 \neq x_1 t$, if f_1 were a function in D taking a value $\neq 1$ at $x_1 t^{-1}$ and the value 1 at all points $x \neq x_1 t^{-1}$, it would follow that $f_1(x_1 t^{-1})^{-1}f_1(x_1) \neq 1$ but $f_1(x_1)^{-1}f_1(x_1 t) = 1$, in violation of the condition. Hence $xt^{-1} = xt$ for all $x \in X$ and therefore $t^2 = 1$. Let $x_1 \in X$. Then $x_1 t \neq x_1$ and since $|X| \geqslant 3$ there is a point $x_2 \in X$ such that $x_2 \neq x_1, x_1 t$. Then $x_2 t \neq x_1, x_1 t, x_2$. Let f_1 be a function in D taking a value $\neq 1$ at x_1 and the value 1 at all points $x \neq x_1$: then $f_1(x_1 t)^{-1}f_1(x_1) \neq 1$ but $f_1(x_2 t)^{-1}f_1(x_2) = 1$, again in violation of the condition. Hence $t = 1$.

Since it is clear that if $t = 1$ and $f(x) \in Z(G)$ for all $x \in X$ then $[tf, f_1] \in Z(W)$ for all $f_1 \in D$, this completes the proof.

Let $f \in Z_2(W)$ and let ζ be the corresponding homomorphism $T \to Z(G)$ such that

$$f(xt) = f(x)\, t^{\zeta} \text{ for all } x \in X \text{ and all } t \in T.$$

Then clearly $\mathrm{Stab}_T(x) \leqslant \operatorname{Ker}\zeta$ for all $x \in X$, and therefore $\overline{\mathrm{Stab}_T(x)} \leqslant \operatorname{Ker}\zeta$ (where $\overline{\mathrm{Stab}_T(x)}$ denotes the *normal closure* of $\mathrm{Stab}_T(x)$ in T, that is, the smallest normal subgroup of T containing $\mathrm{Stab}_T(x)$). Hence if $\overline{\mathrm{Stab}_T(x)} = T$ then ζ is necessarily trivial and so $f \in Z(W)$. Thus Lemma 1 has the

COROLLARY. *With the hypotheses of Lemma* 1, *if* $\overline{\mathrm{Stab}_T(x)} = T$ *for some* $x \in X$ (*and so for all* $x \in X$) *then* $W/Z(W)$ *has trivial centre.*

Note that if $|X| > 2$ there is in Lemma 1 and its Corollary no condition on the group G other than that it is non-trivial. An obvious choice for T is as the symmetric group of degree n permuting in the natural way the set $X = \{1, 2, ..., n\}$, where n is an integer $\geqslant 3$: then the hypotheses of Lemma 1 and the Corollary are satisfied. For the purpose of proving the theorem, however, it is desirable to arrange that if G is a finite ϖ-group, where ϖ is a set of at least two prime numbers, then $W/Z(W)$ is also a finite ϖ-group.

For any two distinct prime numbers, p and q say, the *order of p (mod q)* is the least positive integer s such that $p^s \equiv 1 \pmod q$.

LEMMA 2. *Let p and q be any 2 distinct prime numbers and let s be the order of p (mod q). There is a group T of order $p^s q$ and a permutation representation of T for which all the hypotheses of Lemma* 1 *and its Corollary are fulfilled (for any group $G \neq 1$).*

Proof. There is a non-trivial irreducible representation of degree s of a group Q of order q over the Galois field $\mathrm{GF}(p)$ (defined by mapping a generator of Q to the transformation of the field $\mathrm{GF}(p^s)$ determined by multiplication of $\mathrm{GF}(p^s)$ by an element of $\mathrm{GF}(p^s)^\times$ of order q). Let T be the semi-direct product of an elementary abelian group P of order p^s by Q with action defined by this representation. Since the representation is irreducible, P is a minimal normal subgroup of T, and since also the representation is non-trivial, Q is a non-normal maximal subgroup of T. Let X be the set of right cosets of Q in T. Then T permutes X (by right multiplication) faithfully and transitively. Note that $|X| = p^s > 2$ (for if $p = 2$ then $s > 1$). Finally, $Q \in X$, $\mathrm{Stab}_T(Q) = Q$ and so $\overline{\mathrm{Stab}_T(Q)} = T$.

COROLLARY. *Let ϖ be any set of at least two prime numbers and let G be any finite ϖ-group ($\neq 1$). Then G can be embedded subnormally in a finite ϖ-group $W/Z(W)$ with trivial centre and with a normal subgroup $D/Z(W)$ which is a central product of copies of G and such that W/D is soluble.*

The next step will be to prove

LEMMA 3. *Let ϖ be any set of at least two prime numbers and let H be any finite ϖ-group such that $Z(H) = 1$. Then H can be embedded subnormally in a finite ϖ-group J such that $C_J(H) = 1$,* Out *J is a ϖ'-group and $R(J) \leqslant H$.*

The key to the proof is a recent result of E. Schenkman who gave a new proof of Wielandt's automorphism tower theorem by establishing

LEMMA 4. (E. Schenkman **[6]**). *Let A be a subnormal subgroup of a finite group G such that $C_G(A) = 1$. Then the order of G is bounded in terms of the order of the smallest normal subgroup B of A such that A/B is nilpotent.*

Proof of Lemma 3. From H an ascending sequence of finite groups is constructed in the following way:

$$H = H_0 \leqslant H_1 \leqslant H_2 \leqslant \dots .$$

Suppose $i > 0$ and that $H_0, ..., H_{i-1}$ have been defined with $C_{H_j}(H) = 1$ for each j, and (if $i > 1$) $H_{j-1} \lhd H_j$ and H_j/H_{j-1} is a finite soluble ϖ-group ($j = 1, ..., i-1$). Then $Z(H_{i-1}) = 1$ and so $H_{i-1} \lhd \mathrm{Aut}\, H_{i-1}$. If there is a prime $p \in \varpi$ such that p

divides the order of $\operatorname{Out} H_{i-1}$, let H_i be a subgroup of $\operatorname{Aut} H_{i-1}$ containing H_{i-1} and such that H_i/H_{i-1} has order p; but if $\operatorname{Out} H_{i-1}$ is a ϖ'-group, let $H_i = H_{i-1}$. In any case $H_{i-1} \lhd H_i$ and H_i/H_{i-1} is a finite soluble ϖ-group: moreover, $C_{H_i}(H_{i-1}) = 1$ and therefore, since $C_{H_{i-1}}(H) = 1$, a result of Wielandt [7; (44)] shows that $C_{H_i}(H) = 1$. This completes the definition of the sequence. By Lemma 4, for every i the order of H_i is bounded in terms of the order of H. Therefore the sequence must become stationary in a finite number of steps, say at the term $H_n = J$. By the definition of the sequence, H is subnormal in J, J is a finite ϖ-group, $C_J(H) = 1$ and $\operatorname{Out} J$ is a ϖ'-group. Moreover, since every quotient H_i/H_{i-1} is soluble, $R(J) \leqslant H$.

LEMMA 5. *Let ϖ be any set of at least two prime numbers and suppose that J is a finite ϖ-group such that $Z(J) = 1$ and $\operatorname{Out} J$ is a ϖ'-group. Then there is a complete group K such that $J \leqslant K \leqslant \operatorname{Aut} J$ and $R(K) \leqslant J$.*

Proof. By Lemma 3, with J in place of H and the set of all prime numbers in place of ϖ, J can be embedded subnormally in a finite group K such that $C_K(J) = 1$, $\operatorname{Out} K = 1$ and $R(K) \leqslant J$. In particular, K is complete. Let L be the largest normal ϖ-subgroup of K. Then $J \leqslant L$. Let $N = N_L(J)$. Certainly $C_N(J) = 1$ and so

$$J \leqslant N \leqslant \operatorname{Aut} J.$$

Then N/J is a ϖ-group (since $N \leqslant L$) and N/J is a ϖ'-group (since $\operatorname{Out} J$ is a ϖ'-group). Therefore

$$N = J.$$

Since J is subnormal in K, J is certainly subnormal in L and hence

$$L = J.$$

Therefore

$$J \lhd K.$$

Since also $C_K(J) = 1$, it follows that $J \leqslant K \leqslant \operatorname{Aut} J$.

The Theorem follows at once from the Corollary to Lemma 2, Lemma 3 and Lemma 5.

One may note that in Lemma 5 (and so also in the theorem) K is necessarily a self-normalizing subgroup of $\operatorname{Aut} J$: for, since K is complete, K is a direct factor of $N_{\operatorname{Aut} J}(K)$, say

$$N_{\operatorname{Aut} J}(K) = K \times M,$$

and then since $J \leqslant K$, M consists of automorphisms of J which centralize J; hence $M = 1$. Therefore Lemma 5 has the following

COROLLARY 1. *Let p be a prime number and suppose that J is a finite p'-group such that $Z(J) = 1$ and $\operatorname{Out} J$ is a p-group. Then $\operatorname{Aut} J$ is complete.*

However, it is not true in general that $K = \operatorname{Aut} J$ in Lemma 5. For instance, let $\varpi = \{11, 23\}$, let H be a non-abelian group of order 11×23 (uniquely determined up to isomorphism) and let J be the direct product of 5 copies of H. Then J is a finite ϖ-group and $Z(J) = 1$. Since $Z(J) = 1$, it follows from the Krull–Remak–Schmidt theorem that J has a unique direct decomposition as a product of indecomposable factors (B. Huppert [4; p. 69, Satz I. 12.5]). Therefore since H is indecomposable, every automorphism of J permutes the 5 copies of H in the direct decomposition of J. It follows that Aut J is isomorphic to the natural wreath product of Aut H by Σ_5, the symmetric group of degree 5. Hence Out J is isomorphic to the natural wreath product of Out H by Σ_5. Since Out H has order 2, it follows that Out J has order $2^8 \times 3 \times 5$. Thus Out J is a ϖ'-group. Moreover, since J is soluble and Aut J is insoluble, $K \neq$ Aut J in Lemma 5.

There exist complete finite supersoluble groups: for example, the holomorph of a cyclic group of odd order is complete and supersoluble. But not every finite supersoluble group can be embedded subnormally in a complete finite supersoluble group. In fact it is well known that every finite supersoluble group is 2-nilpotent (see B. Huppert [**4**; p. 716, Satz VI. 9.1]); and it is easy to see that for any prime p, no non-trivial finite p-group can be embedded subnormally in a finite p-nilpotent group with trivial centre; for, if G is a finite p-nilpotent group which contains the non-trivial p-group P as a subnormal subgroup, then P lies in the Fitting subgroup of G and so G has a non-trivial normal p-subgroup; then any minimal normal p-subgroup of G lies in the centre of G.

The final observations of this note concern perfect groups. Suppose that G is a finite perfect group and let K and E be as in the statement of the theorem. Since $R(K) \leqslant E$, $R(K) \lhd E$ and $E/R(K)$ is soluble. However, since E is a central product of copies of G, E is perfect, and so $E = R(K)$. This establishes

COROLLARY 2. *For any finite perfect group G there is a finite complete group K with a normal subgroup E which is a central product of copies of G and such that K/E is soluble.*

Here " central product " cannot be replaced by " direct product ". For instance, let G be the special linear group SL(2,5) of degree 2 over the Galois field GF(5). Let n be any positive integer and let D be the direct product of n copies of G. Since G is perfect and directly indecomposable it follows (B. Huppert [**4**; p. 69, Satz I. 12.5]) that the direct decomposition of D as a product of n copies of G is unique. Let t be the element of order 2 in $Z(G)$ and let z be the element of D whose projection in every copy of G is t. Then z is an element of $Z(D)$ of order 2 and z is fixed by every automorphism of D. Hence z lies in the centre of every group which contains D as a normal subgroup, and so there is no complete group containing D as a normal subgroup.

However, if H is a finite perfect group such that $Z(H) = 1$ then by Lemma 3, H can be embedded subnormally in a finite complete group K such that $C_K(H) = 1$ and $R(K) \leqslant H$. It follows that $H = R(K)$ and so $H \lhd K \leqslant$ Aut H.

COROLLARY 3. *For any finite perfect group H such that* $Z(H) = 1$ *there is a complete group K such that* $H \leqslant K \leqslant$ Aut H *and K/H is soluble.*

This result may be compared with the well-known fact that Aut H is complete whenever H is a direct product of non-abelian simple groups. Of course one would like to know whether, in the special case when H is itself a non-abelian simple group, $K =$ Aut H necessarily: if this could be shown, it would establish a famous conjecture of O. Schreier.

But in Corollary 3, $K \neq \operatorname{Aut} H$ in general. For instance, if G is a finite non-abelian simple group and H is the direct product of n copies of G, where $n \geqslant 5$, then $\operatorname{Aut} H$ is isomorphic to the natural wreath product of $\operatorname{Aut} G$ by Σ_n, the symmetric group of degree n. Then $\operatorname{Out} H$ is insoluble and so $K \neq \operatorname{Aut} H$ in Corollary 3.

In fact there are finite perfect groups H such that $Z(H) = 1$ but $\operatorname{Aut} H$ is not complete. To see this, let G be any finite complete simple group: for example, G can be a Mathieu group of degree 11, 23 or 24 (see for instance N. Burgoyne and P. Fong [**1**]). Then let $H = H_1 \times H_2$, where H_1 is the natural wreath product of G by the alternating group A_5 of degree 5, and H_2 is the direct product of 5 copies of G. Then H is perfect and $Z(H) = 1$. The group H_1 is directly indecomposable and so, by the uniqueness of the direct decomposition of H into indecomposable factors, the subgroups H_1 and H_2 are both characteristic in H. Hence $\operatorname{Aut} H \cong \operatorname{Aut} H_1 \times \operatorname{Aut} H_2$. But $\operatorname{Aut} H_1$ and $\operatorname{Aut} H_2$ are both isomorphic to the natural wreath product W of G by Σ_5. Thus $\operatorname{Aut} H$ is the direct product of 2 copies of W, and so is not complete.

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Department of Pure Mathematics,
The University,
Newcastle upon Tyne.

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UNIVERSAL FINITE GROUP EXTENSIONS AND A NON-SPLITTING THEOREM†

BY

JOHN S. ROSE

ABSTRACT

Let G and K be finite groups whose orders have a common prime divisor. Then there is a group K^* closely related to K for which there is a non-split extension of K^* by G.

In [8] the problem of splitting of group extensions was considered from the following point of view: given a group K and a suitable class $\mathfrak{X}$ of groups, under what circumstances do all extensions of K by $\mathfrak{X}$-groups split? In this note, we make some remarks about the dual question: given a group G and a class $\mathfrak{X}$ of groups, when do all extensions of $\mathfrak{X}$-groups by G split? We discuss only the case in which G is a finite group and $\mathfrak{X}$ a class of finite groups.

A first relevant fact is a result of W. Gaschütz [1]. If G is a finite group and p a prime divisor of the order $|G|$ of G then there is a finite group H with a normal elementary abelian p-subgroup $A \neq 1$ such that $H/A \cong G$ and $A \leqq \Phi(H)$, the Frattini subgroup of H; hence such that H does not split over A. Therefore, if $\mathfrak{X}$ is any class of finite groups which contains all elementary abelian p-groups for all primes p which divide the orders of $\mathfrak{X}$-groups, then a necessary (and of course, by the Schur-Zassenhaus theorem, sufficient) condition for all extensions of $\mathfrak{X}$-groups by G to split is that all $\mathfrak{X}$-groups have orders co-prime to $|G|$. We shall prove the following sharper non-splitting result:

THEOREM 1. *Let G and K be finite groups such that $(|G|,|K|) > 1$. Then there is a non-split extension of a group K^* by G, where K^* is a subgroup of*

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a finite direct product of copies of K, *and* K *is an epimorphic image of* K^*. *In particular, for every prime* p, *the Sylow* p-*subgroups of* K *and* K^* *have the same class, derived length and exponent; and if* K *is soluble,* K *and* K^* *have the same derived length, nilpotent length and* p-*length for all primes* p.

On the other hand, we cannot in general choose K^* in Theorem 1 to be a direct product of copies of K, in view of the following simple fact.

THEOREM 2. *The class of finite groups all extensions of which split is closed under the formation of finite direct products.*

This class certainly contains non-trivial groups since it contains for instance all complete finite groups; it also contains groups which are not complete: see [8].

In order to prove Theorem 1, we need a straightforward generalization of a fundamental result of Gaschütz [1]. Before stating this, we introduce some notation and terminology. We use P. Hall's convenient notion of closure operations on classes of groups: see [4, §1.3]. Thus a class $\mathfrak{X}$ of groups is said to be S-closed if every subgroup of an $\mathfrak{X}$-group is an $\mathfrak{X}$-group, D_0-closed if the direct product of any 2 $\mathfrak{X}$-groups is an $\mathfrak{X}$-group, and R_0-closed if any subdirect product of 2 $\mathfrak{X}$-groups is an $\mathfrak{X}$-group. If t is a positive integer, a group K is said to be a t-generator group if K has a generating set of elements with at most t members. We shall call a class $\mathfrak{X}$ of finite groups *bounded* if, for every positive integer t, there is a corresponding positive integer $X(t)$ such that all t-generator $\mathfrak{X}$-groups have orders $\leqq X(t)$.

The generalization of Gaschütz's result which we shall use is

THEOREM 3. *Let* n *be a positive integer,* G *a finite* n-*generator group and* $\mathfrak{X}$ *a bounded* R_0-*closed* class *of finite groups. Let* $\mathscr{C}$ *be the class of all group extensions* $1 \to K \xrightarrow{\iota} H \to G \to 1$ (*where* ι *denotes the inclusion map*) *such that* K *is an* $\mathfrak{X}$-*group and* H *an* n-*generator group. Then there is in* $\mathscr{C}$ *an extension* $1 \to K^* \to H^* \to G \to 1$ *which is universal for* $\mathscr{C}$ *in the following sense. For any extension* $1 \to K \to H \to G \to 1$ *in* $\mathscr{C}$ *there is an epimorphism* $\chi: H^* \to H$ *such that the diagram below is commutative.*

$$\begin{tikzcd}
& K^* \arrow[r] \arrow[d] & H^* \arrow[dr] \arrow[d, "\chi"] & & \\
1 \arrow[ur] \arrow[dr] & & & G \arrow[r] & 1 \\
& K \arrow[r] & H \arrow[ur] & &
\end{tikzcd}$$

Fig. 1

In this connexion, see the remarks at the beginning of Chap. 9 of K. W. Gruenberg's book [3].

REMARK. In the diagram above, since χ is an epimorphism, χ maps K^* onto K.

In order to prove Theorem 3 we first prove

LEMMA 1. *Let n be a positive integer and F a free group of rank n. Let G and H be n-generator groups for which there is an epimorphism $\zeta: H \to G$ such that* $\operatorname{Ker} \zeta$ *is finite. Then, for any epimorphism $\theta: F \to G$, there is an epimorphism $\eta: F \to H$ such that*

$$\theta = \eta\zeta.$$

The appropriate diagram is

$$\begin{tikzcd}
F \arrow[rr, "\theta"] \arrow[dr, dashed, "\eta"'] & & G \\
& H \arrow[ur, "\zeta"'] &
\end{tikzcd}$$

Fig. 2

PROOF. Let $\{x_1, \cdots, x_n\}$ be a set of free generators of F and let

$$x_j\theta = g_j \text{ for } j = 1, \cdots, n.$$

Then

$$G = \langle g_1, \cdots, g_n \rangle.$$

Since $\operatorname{Ker} \zeta$ is finite and H is an n-generator group, it follows from a result of Gaschütz ([2, Satz 1]) that there is a set $\{h_1, \cdots, h_n\}$ of generators of H such that

$$h_j\zeta = g_j \quad \text{for } j = 1, \cdots, n.$$

Now there is a (unique) homomorphism $\eta: F \to H$ such that

$$x_j\eta = h_j \quad \text{for } j = 1, \cdots, n.$$

Since $H = \langle h_1, \cdots, h_n \rangle$ and $F = \langle x_1, \cdots, x_n \rangle$, it follows that η is an epimorphism and

$$\eta\zeta = \theta.$$

PROOF OF THEOREM 3. Let F be a free group of rank n and let

$$1 \to R \to F \xrightarrow{\theta} G \to 1$$

be a presentation of G. By Schreier's theorem, R is finitely generated. Therefore, since $\mathfrak{X}$ is a bounded class, the quotient groups of R which are $\mathfrak{X}$-groups have bounded orders. We choose a normal subgroup T of R such that R/T is an $\mathfrak{X}$-group of maximal order. Then, since $\mathfrak{X}$ is R_0-closed, T is in fact the unique smallest normal subgroup of R with an $\mathfrak{X}$-group as quotient. Hence T is normal in F.

Let $\bar{\theta}$ be the homomorphism of F/T onto G induced by θ. Then

$$1 \to R/T \to F/T \xrightarrow{\bar{\theta}} G \to 1$$

is an extension in the class $\mathscr{C}$. We claim that it is universal for $\mathscr{C}$ in the sense defined.

Let

$$1 \to K \to H \xrightarrow{\zeta} G \to 1$$

be any extension in $\mathscr{C}$. Since G and H are n-generator groups and $\operatorname{Ker}\zeta = K$, which is finite, we can apply Lemma 1. This guarantees the existence of a presentation of H, say

$$1 \to S \to F \xrightarrow{\eta} H \to 1$$

such that

$$\theta = \eta\zeta.$$

Then

$$S = \operatorname{Ker}\eta \leqq \operatorname{Ker}\theta = R.$$

Moreover, η induces an isomorphism of F/S onto H in which R/S is mapped to $\operatorname{Ker}\zeta = K$. Therefore R/S is an $\mathfrak{X}$-group, and so

$$T \leqq S.$$

Hence η induces an epimorphism $\bar{\eta}\colon F/T \to H$ such that

$$\bar{\eta}\zeta = \bar{\theta}.$$

Moreover, $\bar{\eta}$ maps R/T to K. Hence we have a commutative diagram

$$\begin{tikzcd}
 & R/T \arrow[r] \arrow[dd] & F/T \arrow[dd, "\bar{\eta}"] \arrow[dr, "\bar{\theta}"] & & \\
1 \arrow[ur] \arrow[dr] & & & G \arrow[r] & 1 \\
 & K \arrow[r] & H \arrow[ur, "\zeta"'] & &
\end{tikzcd}$$

Fig. 3

as required.

As particular choices for $\mathfrak{X}$ in Theorem 3 we may take

(i) for any positive integers m and s, $\mathfrak{X}$ = the class of finite soluble groups of exponents dividing m and derived lengths $\leqq s$;

(ii) for any prime p, $\mathfrak{X}$ = the class of finite groups of exponent p (and 1): this by a famous theorem of A. I. Kostrikin [6].

Gaschütz's original result ([1, Satz 1]) corresponds to choosing $m = p$, a prime, and $s = 1$ in (i). Now in order to prove Theorem 1 we shall show that another possible choice for $\mathfrak{X}$ in Theorem 3 is

(iii) for any finite group K, $\mathfrak{X}$ = the smallest $\{\mathrm{S}, \mathrm{D}_0\}$-closed class of groups containing K.

Since a class of groups which is $\{\mathrm{S}, \mathrm{D}_0\}$-closed is certainly R_0-closed, what we have to show is that the smallest $\{\mathrm{S}, \mathrm{D}_0\}$-closed class of groups containing any finite group K is a bounded class. This is the content of Lemma 2.

LEMMA 2. *Let K be a finite group. Then the class of all subgroups of finite direct products of copies of K is a bounded class of groups.*

This follows from Theorem 15.71 of H. Neumann's book [7]. A direct proof is included here.

PROOF. We show that for every positive integer t, every t-generator subgroup H of a finite direct product of copies of K has order $\leqq |K|^{|K|^t}$. Let $H \leqq K_1 \times \cdots \times K_n$, where n is a positive integer and each K_j is a copy of $K (j = 1, \cdots, n)$. We argue by induction on n. The assertion is trivial for $n = 1$, so we suppose that $n > 1$. By the induction hypothesis we may assume that H is not isomorphic to a subgroup of a direct product of $n - 1$ copies of K. Let F be a free group of rank t and let

$$1 \to R \to F \xrightarrow{\theta} H \to 1$$

be a presentation of H. For $j = 1, \cdots, n$ let π_j be the projection homomorphism of $K_1 \times \cdots \times K_n$ onto K which maps each element of $K_1 \times \cdots \times K_n$ onto its jth component; and let ι denote the inclusion map of H in $K_1 \times \cdots \times K_n$. Then $\theta\iota\pi_1, \cdots, \theta\iota\pi_n$ are homomorphisms of F into K. Now if $\theta\iota\pi_r = \theta\iota\pi_s$ with $1 \leqq r < s \leqq n$ then, since θ maps F onto H, every element of H would have its rth and sth components equal; but then H would be isomorphic to a subgroup of a direct product of $n - 1$ copies of K, contrary to assumption. Therefore $\theta\iota\pi_1, \cdots, \theta\iota\pi_n$ are distinct homomorphisms of F into K. But since each homomorphism of F into K is determined by its effect on a set of t free generators of F, there are just $|K|^t$ distinct homomorphisms of F into K. Hence $n \leqq |K|^t$ and so

$$|H| \leqq |K|^n \leqq |K|^{|K|^t}.$$

This completes the induction proof.

We use also

LEMMA 3. *Let $\mathfrak{X}$ be any bounded, $\{\mathrm{s}, \mathrm{D}_0\}$-closed class of finite groups. Let n be a positive integer, G a finite n-generator group and $\mathscr{C}$ the class of extensions defined in Theorem 3. Let $1 \to K^* \to H^* \to G \to 1$ be an extension in $\mathscr{C}$ which is universal for $\mathscr{C}$. Then H^* splits over K^* if and only if all $\mathfrak{X}$-groups have orders co-prime to $|G|$.*

PROOF. If $(|G|, |K^*|) = 1$ then H^* splits over K^*, by the Schur-Zassenhaus theorem.

Now suppose that there is an $\mathfrak{X}$-group J such that $(|G|, |J|) > 1$. Let p be a common prime divisor of $|G|$ and $|J|$. Since $\mathfrak{X}$ is $\{\mathrm{s}, \mathrm{D}_0\}$-closed, $\mathfrak{X}$ contains all finite elementary abelian p-groups. By a result of Gaschütz [1] mentioned above, there is an extension

$$1 \to A \to H \to G \to 1,$$

where A is an elementary abelian p-group and $1 < A \leqq \Phi(H)$. It follows from this, since G is n-generator, that H is n-generator, and therefore that the extension belongs to $\mathscr{C}$. Hence there is an epimorphism $\chi: H^* \to H$ making a commutative diagram.

$$\begin{tikzcd}
 & K^* \arrow[r] \arrow[dd] & H^* \arrow[dr] \arrow[dd, "\chi"] & & \\
1 \arrow[ur] \arrow[dr] & & & G \arrow[r] & 1 \\
 & A \arrow[r] & H \arrow[ur] & &
\end{tikzcd}$$

Fig. 4

Then, since $\operatorname{Ker}\chi \leqq K^*$, if H^* were to split over K^* it would follow that H split over A, which is false. Hence H^* does not split over K^*

PROOF OF THEOREM 1. We suppose that G and K are finite groups such that $(|G|, |K|) > 1$. Let $\mathfrak{X}$ be the class of all subgroups of finite direct products of copies of K; thus $\mathfrak{X}$ is the smallest $\{\mathrm{s}, \mathrm{D}_0\}$-closed class of groups containing K. By Lemma 2, $\mathfrak{X}$ is a bounded class. Since also $\mathfrak{X}$ is R_0-closed, Theorem 3 is applicable. Let n be a positive integer such that the direct product $G \times K$ is an n-generator group and let $\mathscr{C}$ be the class of extensions defined in Theorem 3, with $\mathfrak{X}$ as above. Let

$$1 \to K^* \to H^* \to G \to 1$$

be an extension in $\mathscr{C}$ which is universal for $\mathscr{C}$. There is also in $\mathscr{C}$ an extension

$$1 \to K \to G \times K \to G \to 1 .$$

Hence there is a commutative diagram

$$\begin{tikzcd}
& K^* \arrow[r] \arrow[dd] & H^* \arrow[dr] \arrow[dd] & & \\
1 \arrow[ur] \arrow[dr] & & & G \arrow[r] & 1 \\
& K \arrow[r] & G \times K \arrow[ur] & &
\end{tikzcd}$$

Fig. 5

in which K^* is mapped onto K. Thus K is an epimorphic image of K^*, which is a subgroup of a finite direct product of copies of K. Moreover,

$$1 \to K^* \to H^* \to G \to 1$$

is an extension of K^* by G which, by Lemma 3, does not split.

To prove Theorem 2, we note first

LEMMA 4. *Let n be a positive integer and let $L_1, \cdots, L_n$ be normal subgroups of, respectively, groups $K_1, \cdots, K_n$. Then the direct product $K_1 \times \cdots \times K_n$ splits over $L_1 \times \cdots \times L_n$ if and only if each K_j splits over L_j, for $j = 1, \cdots, n$.*

PROOF. Let $K = K_1 \times \cdots \times K_n$ and $L = L_1 \times \cdots \times L_n$. If J_j is a complement to L_j in K_j for $j = 1, \cdots, n$ then clearly $J_1 \times \cdots \times J_n$ is a complement to L in K. Conversely, if J is a complement to L in K then, for $j = 1, \cdots, n$, $(JL^j) \cap K_j$ is a complement to L_j in K_j, where L^j is the product of all the L_i's except L_j.

LEMMA 5. *Let K be a group with a normal subgroup L, and let n be a positive integer. Let W denote the natural wreath product of K by Σ_n, the symmetric group of degree n, let $K_1 \times \cdots \times K_n$ denote the base group of W (a direct product of n copies of K) and let $L_1 \times \cdots \times L_n$ denote the corresponding direct product of n copies of L, which is a normal subgroup of W. Then W splits over $L_1 \times \cdots \times L_n$ if and only if K splits over L.*

PROOF. If W splits over $L_1 \times \cdots \times L_n$ then $K_1 \times \cdots \times K_n$ splits over $L_1 \times \cdots \times L_n$, and so, by Lemma 4, K splits over L.

Conversely, suppose that K splits over L, and let J be a complement to L in K. Let $J_1 \times \cdots \times J_n$ denote the corresponding direct product of n copies of J which is a subgroup of W normalized by Σ_n. Now it is clear that $(J_1 \times \cdots \times J_n)\Sigma_n$ is a subgroup of W which is a complement to $L_1 \times \cdots \times L_n$ in W.

Now let $\mathfrak{Y}$ denote the class of finite groups all extensions of which split. A finite group K is a $\mathfrak{Y}$-group if and only if $Z(K) = 1$ and $\operatorname{Aut} K$ splits over $\operatorname{Inn} K$: see [8, Corollary 2.3].

LEMMA 6. *Let K be any non-trivial $\mathfrak{Y}$-group. Then any indecomposable direct factor of K is also a $\mathfrak{Y}$-group.*

PROOF. Say $K = K_{11} \times \cdots \times K_{1r_1} \times K_{21} \times \cdots \times K_{2r_2} \times \cdots \times K_{s1} \times \cdots \times K_{sr_s}$ where $s, r_1, r_2, \cdots, r_s$ are positive integers, each K_{ij} is a directly indecomposable group and $K_{ij} \cong K_{i'j'}$ if and only if $i = i'$, for $1 \leqq i,\ i' \leqq s,\ 1 \leqq j \leqq r_i$, $1 \leqq j' \leqq r_{i'}$. Since by hypothesis $Z(K) = 1$, it follows that $Z(K_{ij}) = 1$ for all i,j. Also, by the Krull-Remak-Schmidt theorem ([5, I.12.6]) the decomposition of K above is the unique decomposition of K as a direct product of indecomposable factors. Hence, for $i = 1, \cdots, s$, $K_{i1} \times \cdots \times K_{ir_i}$ is a characteristic subgroup of K; and

$$\operatorname{Aut} K \cong W_1 \times W_2 \times \cdots \times W_s,$$

where, for $i = 1, \cdots, s, W_i$ is the natural wreath product of $\operatorname{Aut} K_{i1}$ by Σ_{r_i}. The normal subgroup of $W_1 \times \cdots \times W_s$ corresponding to $\operatorname{Inn} K$ is $Y_1 \times \cdots \times Y_s$, where, for $i = 1, \cdots, s, Y_i$ is the direct product of r_i copies of $\operatorname{Inn} K_{i1}$ naturally contained in the base group of W_i. By hypothesis, $\operatorname{Aut} K$ splits over $\operatorname{Inn} K$. Hence, by Lemma 4, W_i splits over Y_i, for $i = 1, \cdots, s$; then, also by Lemma 5, $\operatorname{Aut} K_{i1}$ splits over $\operatorname{Inn} K_{i1}$. Hence, for $i = 1, \cdots, s$, K_{i1} is a $\mathfrak{Y}$-group. This proves the lemma.

PROOF OF THEOREM 2. We have to show that if K_1 and K_2 are $\mathfrak{Y}$-groups then the direct product $K_1 \times K_2$ is a $\mathfrak{Y}$-group. Each of K_1 and K_2 can be expressed (by the Krull-Remak-Schmidt theorem uniquely) as a direct product of indecomposable factors; and by Lemma 6, these indecomposable factors are also $\mathfrak{Y}$-groups. Hence, in order to prove Theorem 2, it is enough to show that any finite direct product of directly indecomposable $\mathfrak{Y}$-groups is a $\mathfrak{Y}$-group.

Let $K = K_{11} \times \cdots \times K_{1r_1} \times K_{21} \times \cdots \times K_{2r_2} \times \cdots \times K_{s1} \times \cdots \times K_{sr_s}$, where $s, r_1, r_2, \cdots, r_s$ are positive integers, each K_{ij} is a directly indecomposable $\mathfrak{Y}$-group and $K_{ij} \cong K_{i'j'}$ if and only if $i = i'$, for $1 \leqq i, i' \leqq s, 1 \leqq j \leqq r_i, 1 \leqq j' \leqq r_{i'}$. Since $Z(K_{ij}) = 1$ for all i, j, $Z(K) = 1$. Also, as in the proof of Lemma 6,

$$\mathrm{Aut}\, K \cong W_1 \times W_2 \times \cdots \times W_s,$$

where for $i = 1, \cdots, s$, W_i is the natural wreath product of $\mathrm{Aut}\, K_{i1}$ by Σ_{r_i}. As before, let the subgroup of $W_1 \times \cdots \times W_s$ corresponding to $\mathrm{Inn}\, K$ be $Y_1 \times \cdots \times Y_s$, where Y_i is the direct product of r_i copies of $\mathrm{Inn}\, K_{i1}$. Since $\mathrm{Aut}\, K_{i1}$ splits over $\mathrm{Inn}\, K_{i1}$, Lemma 5 shows that W_i splits over Y_i. Then by Lemma 4, $W_1 \times \cdots \times W$ splits over $Y_1 \times \cdots \times Y_s$. Hence K is a $\mathfrak{Y}$-group, as required.

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Sufficient Conditions for the Existence of Ordered Sylow Towers in Finite Groups

JOHN S. ROSE

Department of Pure Mathematics, The University, Newcastle upon Tyne, England

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Throughout this paper, let G denote a finite group and p a prime number. Notation used without further explanation is standard. If G is p-soluble, $r_p(G)$ will denote the largest integer n such that G has a chief factor of order p^n (with $r_p(G) = 0$ if p does not divide $|G|$): see [7, p. 685, VI.5.2]. If G is soluble and $r_p(G) = 1$ for every prime divisor p of $|G|$ then G is called *supersoluble*. A familiar consequence of the supersolubility of G is that G possesses an *ordered Sylow tower*, that is to say there is a series $1 = G_0 < G_1 < \cdots < G_s = G$ of normal subgroups of G such that for each $i = 1, \ldots, s$, G_i/G_{i-1} is isomorphic to a Sylow p_i-subgroup of G, where $p_1, \ldots, p_s$ are the distinct prime divisors of $|G|$ and $p_1 > p_2 > \cdots > p_s$ [7, p. 715, VI.9.1]. As a partial generalization of this result, B. Huppert [6, Satz 14]; see also [7, VI.9.1]) proved that if G is a soluble group such that $r_p(G) \leqslant 2$ for every prime divisor p of $|G|$, and if $|G|$ is not divisible by 2 or 3, then G possesses an ordered Sylow tower. More recently, K. A. Corradi [2] showed that a group G possesses an ordered Sylow tower if $|G|$ is divisible neither by 12 nor by the cube of any prime except perhaps its largest prime divisor. It will be shown here that the only difficulty which would arise by allowing the primes 2 and 3 to appear in these results arises from the possible involvement in G of the alternating group A^4 of degree 4. We recall that a group Q is said to be *involved* in G if there is a subgroup H of G and a normal subgroup K of H such that H/K is isomorphic to Q.

Specifically we shall prove the following results.

THEOREM A. *Let G be p-soluble, where p is the smallest prime divisor of $|G|$. If $r_p(G) \leqslant 2$ then either G is p-nilpotent or $p = 2$ and A^4 is involved in G.*

(It follows from the first hypothesis on G and the Feit–Thompson Theorem that G is actually soluble; but we shall not make use of this.)

An immediate consequence is the following generalization of Huppert's result.

COROLLARY 1. *Let G be soluble and suppose that $r_p(G) \leqslant 2$ for every prime divisor p of $|G|$ except perhaps the largest. Then either G possesses an ordered Sylow tower or A^4 is involved in G.*

COROLLARY 2. *Suppose that every p-subgroup of G can be generated by 2 elements, where p is the smallest prime divisor of $|G|$. Then either G is p-nilpotent or $p = 2$ and A^4 is involved in G.*

Here it is not assumed that G is p-soluble. This result generalizes [7, p. 437, VI.5.11]. When $p = 2$, A. R. Camina and T. M. Gagen [1] have obtained a much stronger result in case G has a Sylow 2-subgroup S with a cyclic normal subgroup N such that S/N is cyclic of order >2. They showed that then G contains a 2-nilpotent normal subgroup of index a divisor of 6.

The following generalization of Corradi's result follows at once from Corollary 2.

COROLLARY 3. *Suppose that every p-subgroup of G can be generated by 2 elements for every prime divisor p of $|G|$ except perhaps the largest. Then either G possesses an ordered Sylow tower or A^4 is involved in G.*

For $p > 2$, a better result than Corollary 2 is known. If G contains no elementary abelian subgroup of order p^3, where p is the smallest prime divisor of $|G|$ and $|G|$ is odd, then G is p-nilpotent (W. Feit and J. G. Thompson [3, Lemma 8.5]; see also [4, p. 257, 7.6.1]). It is also known that for $p > 2$, a p-group P has no elementary abelian subgroup of order p^3 if (and only if) all its abelian normal subgroups can be generated by 2 elements (W. Feit and J. G. Thompson [3, Lemma 8.4] and [4, p. 202, 5.4.15]). The corresponding statement for $p = 2$ is false: the situation in that case has been investigated by Anne R. MacWilliams [9]. Here we shall establish a result which generalizes the p-nilpotency result.

THEOREM B. *Suppose that G has no elementary abelian subgroup of order p^3, for some prime divisor p of $|G|$. Then one of the following statements holds:*

(a) *G is p-nilpotent, or*

(b) *$p^2 \equiv 1(q)$ for some prime divisor q of $|G|$, or*

(c) *$p = 2$ and $|G|$ is divisible by 5.*

In their proof, Feit and Thompson make use of the fact that if P is a

p-group with no elementary abelian subgroup of order p^3, $p > 2$ and P has an automorphism of prime order $q \neq p$ then $p^2 \equiv 1(q)$ ([3, Lemma 8.4] and [4, p. 202, 5.4.15]). Here the procedure is reversed and we deduce from Theorem B

COROLLARY 4. *Let P be a p-group which has no elementary abelian subgroup of order p^3, and suppose that P has an automorphism of prime order $q \neq p$. Then either $p^2 \equiv 1(q)$ or $p = 2$ and $q = 5$.*

In fact we shall prove slightly more explicit results than Theorems A and B. In order to state these we need to introduce some notation.

Let T denote the central product of a dihedral group of order 8 and a quaternion group of order 8 obtained by identifying their centres. Then T is an extra-special group of order 2^5. We shall need the following facts about T.

LEMMA 1. (i) *T has no elementary abelian subgroup of order 2^3 and T possesses automorphisms of order 5.*

(ii) *An extra-special group of order 2^5 with an automorphism of order 5 is necessarily isomorphic to T.*

(iii) *There is just one isomorphism type of semi-direct product of T by a group of order 5 with non-trivial action.*

Proof. (i) Since $T' = Z(T)$, of order 2, an elementary abelian subgroup A of T of greatest possible order must be normal in T. Since any maximal abelian normal subgroup of T can be generated by 2 elements [7, p. 355, III. 13.8], $|A| = 2^2$.

To see that T possesses automorphisms of order 5, we may refer to [7, p. 357, III. 13.9b]. Explicitly, we can define an automorphism of T of order 5 as follows. We have

$$T = \langle x, t, u, v \rangle,$$

where $x^4 = t^2 = 1$, $x^2 = u^2 = v^2$, $xt = tx^{-1}$, $uv = vu^{-1}$ and $\langle x, t \rangle$ and $\langle u, v \rangle$ centralize each other. Then it can be checked directly that there is a unique automorphism α of T with

$$x^\alpha = uv, \qquad t^\alpha = xu, \qquad u^\alpha = x^{-1}, \qquad v^\alpha = tuv,$$

and α has order 5.

(ii) An extra-special group of order 2^5 which is not isomorphic to T is necessarily isomorphic to the central product, X say, of two dihedral groups of order 8 obtained by identifying their centres [7, III.13.8]. By reference to [7, III.13.9b and p. 248, II.10.16d] we see that X has no automorphism of order 5; explicitly, we can argue as follows. Let $\hat{X} = X/X'$ and let the usual "circumflex convention" apply. There is a quadratic form $q: \hat{X} \to \mathrm{GF}(2)$ on the vector space $\hat{X}$, defined by

$$q(\hat{y}) = \begin{cases} 0 & \text{if } y^2 = 1, \\ 1 & \text{if } y^2 \neq 1 \end{cases} \qquad y \in X$$

([7, III.13.8c]). Any automorphism of X induces an orthogonal transformation of $\hat{X}$ corresponding to q.

Now

$$X = \langle x_1, t_1, x_2, t_2 \rangle,$$

where $x_i^4 = t_i^2 = (x_i t_i)^2 = 1$ for $i = 1, 2$, $x_1^2 = x_2^2$ and $\langle x_1, t_1 \rangle$ and $\langle x_2, t_2 \rangle$ centralize each other. The elements of $\hat{X}$ on which q takes the value 1 are just

$$\widehat{x_1}, \widehat{x_1 t_2}, \widehat{x_1 x_2 t_2}, \widehat{x_2}, \widehat{t_1 x_2}, \widehat{x_1 t_1 x_2};$$

on the other 10 elements of $\hat{X}$, q takes the value 0. If X had an automorphism α of order 5, the corresponding orthogonal transformation $\hat{\alpha}$ of $\hat{X}$ would also have order 5 (by a well-known theorem of Burnside, since $X' = \Phi(X)$: see [7, p. 275, III.3.18], [4, p. 174, 5.1.4]). Since $\hat{\alpha}$ would permute the 6 nontrivial elements of $\hat{X}$ on which q takes the value 1, $\hat{\alpha}$ would necessarily fix one of these elements. But then by Maschke's theorem, the order of $\hat{\alpha}$ would be a divisor of $|\mathrm{GL}(3, 2)| = 168$, a contradiction.

(iii) Any automorphism of T induces naturally an automorphism of T/T'. This correspondence defines a homomorphism of Aut T into $\mathrm{GL}(4, 2)$, and by the same theorem of Burnside as before, the kernel of this homomorphism is a 2-group. Therefore, since the Sylow 5-subgroups of $\mathrm{GL}(4, 2)$ have order 5, the Sylow 5-subgroups of Aut T have order 5.

Now if G is any semidirect product of T by a group of order 5 with nontrivial action, the action is in fact faithful. Let Hol T denote the holomorph of T. Then G is isomorphic to subgroup of Hol T of the form TH with $|H| = 5$. All such subgroups are conjugate in Hol T because all subgroups of Aut T of order 5 are conjugate in Aut T. Hence any two semidirect products of T by groups of order 5 with nontrivial actions are isomorphic.

Let B^{32} denote a semidirect product of T by a group of order 5 with nontrivial action. (The superscript 32 is used by analogy with the superscript 4 in A^4, for one can show that 32 is the minimal degree of a faithful permutation representation of B^{32}. Of course, there is a transitive such representation on the cosets of a subgroup of B^{32} of order 5.)

If q is a prime number such that $p \equiv 1(q)$ we let $C_{p,q}$ denote a nonabelian group of order pq. There is just one isomorphism type of such groups. If q is an odd prime number such that $p \equiv -1(q)$ we let $D_{p^2,q}$ denote the semidirect product of the additive group of the field $\mathrm{GF}(p^2)$ by a group of order q, with action determined by multiplication in $\mathrm{GF}(p^2)$ by an element of order q in the multiplicative group of $\mathrm{GF}(p^2)$. We note

LEMMA 2. *Let q be an odd prime number such that $p \equiv -1(q)$. If p is odd, there is a unique isomorphism type of nonabelian groups of order p^2q (and $D_{p^2,q}$ is such a group). Also $D_{2^2,3}$ is isomorphic to A^4.*

Proof. Suppose that p is odd and let G be a nonabelian group of order p^2q and P a Sylow p-subgroup of G. Let n denote the number of Sylow p-subgroups of G. Then $n \equiv 1(p)$ and n divides q. Hence if $n \neq 1$, $n = q$ and so $q \equiv 1(p)$ and $p \equiv -1(q)$: then $q > p$ and $p + 1 \geqslant q$, hence $p = 2$, contrary to hypothesis. Therefore P is normal in G and G is a semidirect product of P by a group of order q. The action is nontrivial and therefore in this case must be faithful. Hence G is isomorphic to a subgroup of Hol P, the holomorph of P, of the form PQ, where Q is a subgroup of order q. Also $|P| = p^2$ and since $p \not\equiv 1(q)$, P must be elementary abelian. Hence Aut P is isomorphic to $\mathrm{GL}(2, p)$. Since $\mathrm{GL}(2, p)$ has a cyclic subgroup of order $p^2 - 1$ and since q does not divide $p^2 - p$ and $|\mathrm{GL}(2, p)| = (p^2 - 1)(p^2 - p)$, Aut P has cyclic Sylow q-subgroups. Hence the subgroups of Aut P of order q are all conjugate in Aut P. Therefore all the subgroups of Hol P of the form PQ, with $|Q| = q$, are conjugate in Hol P and therefore isomorphic.

It is clear that $D_{2^2,3}$ is isomorphic to A^4.

The results to be proved are the following generalizations of Theorems A and B.

THEOREM A′. *Let G be a p-soluble with $r_p(G) \leqslant 2$. If G is not p-nilpotent then for some prime divisor q of $|G|$, either $p \equiv 1(q)$ and $C_{p,q}$ is involved in G or q is odd, $p \equiv -1(q)$ and $D_{p^2,q}$ is involved in G.*

THEOREM B′. *Suppose that G has no elementary abelian subgroup of order p^3, for some prime divisor p of $|G|$. Then one of the following statements holds:*

(i) *G is p-nilpotent, or*

(ii) *$p > 2$ and for some prime divisor q of $|G|$ either $p \equiv 1(q)$ and $C_{p,q}$ is involved in G or q is odd, $p \equiv -1(q)$ and $D_{p^2,q}$ is involved in G, or*

(iii) *$p = 2$ and either A^4 or B^{32} is involved in G.*

In connexion with Theorems A′ and B′ we note that J. G. Thompson has proved that if $p > 2$ and G is p-soluble with no elementary abelian subgroup of order p^3 then in fact $r_p(G) \leqslant 2$ ([11, Lemma 5.2.4]).

From Theorem B′ another sufficient condition for the existence of ordered Sylow towers follows at once.

COROLLARY 5. *Suppose that for every prime divisor p of $|G|$ except perhaps the largest, G has no elementary abelian subgroup of order p^3. Then if G does not possess an ordered Sylow tower, either A^4 or B^{32} is involved in G.*

We prove Theorems A′ and B′ by least counter-example arguments. In either case suppose the result false and let G denote a counter-example of least possible order. The hypotheses are inherited by all subgroups of G, so we may assume that G is a group which is not p-nilpotent but whose proper subgroups are all p-nilpotent. Now a theorem of N. Itô [8; and 7, p. 434, IV.5.4] is applicable. G must be a minimal nonnilpotent group, with normal Sylow p-subgroup P and G/P a cyclic q-group for some prime $q \neq p$.

We shall make use of standard properties of minimal nonnilpotent groups, to be found in [7, p. 281, III.5.2]. In addition we need the following facts, which are contained in a paper of L. Rédei [10; see also 5, especially pp. 16–17]. Short proofs are included here.

LEMMA 3. *Let G be a minimal nonnilpotent group, so that G has a normal Sylow p-subgroup P for some prime p and G/P is a cyclic q-group for some prime $q \neq p$. Then P/P' is a chief factor of G, P' is elementary abelian and $|P/P'| = p^{o(p,q)}$, where $o(p, q)$ denotes the least positive integer n such that $p^n \equiv 1(q)$.*

Proof. Let Q be a Sylow q-subgroup of G. We note first that $P/\Phi(P)$ is a chief factor of G. If not there would be a normal subgroup P_1 of G with $\Phi(P) < P_1 < P$, and then by Maschke's theorem there would also be a normal subgroup P_2 of G with $\Phi(P) < P_2 < P$ and

$$P/\Phi(P) = P_1/\Phi(P) \times P_2/\Phi(P).$$

But then P_1Q and P_2Q would be proper subgroups of G and therefore nilpotent. From this it would follow that $C_G(Q) \geqslant P_1P_2 = P$. This is false since G is not nilpotent.

Next, P' is abelian (since by [7, III.5.2b], $P' \leqslant Z(G)$) and has exponent p or 1. The justification for the latter assertion follows the argument for [7, III.5.2c]: namely, since P has class $\leqslant 2$,

$$[x_1, x_2]^p = [x_1{}^p, x_2]$$

for any elements x_1, x_2 of P; hence since $x_1^p \in \Phi(P) \leqslant Z(G)$,

$$[x_1, x_2]^p = 1.$$

Thus the abelian group P' is generated by elements of orders dividing p and therefore P' has exponent dividing p.

Now let $\hat{G} = G/P'$ (with the "circumflex convention" applying). Since $P' \leqslant Z(G)$, $\hat{G}$ is nonnilpotent. We shall prove that $P' = \Phi(P)$ by showing that $\hat{P}$ has exponent p, that is (since $\hat{P}$ is abelian) by showing that $\Omega_1(\hat{P}) = \hat{P}$. Suppose to the contrary that $\Omega_1(\hat{P}) < \hat{P}$. Then $\Omega_1(\hat{P})\,\Phi(\hat{P}) < \hat{P}$ and therefore, since $\Phi(\hat{P}) = \widehat{\Phi(P)}$ and hence by what we have proved $\hat{P}/\Phi(\hat{P})$ is a chief factor of $\hat{G}$, $\Omega_1(\hat{P}) \leqslant \Phi(\hat{P})$. But then $\hat{Q}$ centralizes $\Omega_1(\hat{P})$ and therefore [4, p. 178, 5.2.4] $\hat{Q}$ centralizes $\hat{P}$. But this is contrary to the fact that $\hat{G}$ is nonnilpotent. Hence $P' = \Phi(P)$, as claimed.

Finally, let $|P/P'| = p^n$. Since $\Phi(Q) \leqslant Z(G)$, $G/\Phi(Q)$ is nonnilpotent. Hence the group $Q/\Phi(Q)$, of order q, acts faithfully and irreducibly on P/P'. Now it follows from [7, p. 165, II.3.10] that $n = o(p, q)$.

Proof of Theorem A′. Suppose the result false and let G be a counterexample of least possible order. Then as pointed out above, G is a minimal nonnilpotent group with normal Sylow p-subgroup P and G/P is a cyclic q-group for some prime $q \neq p$. In fact, $|G/P| = q$ since otherwise G would have a nontrivial normal q-subgroup, and then the quotient of G by this would be p-nilpotent, since the hypotheses on G are obviously inherited by quotients of G. But then G would itself be p-nilpotent, contrary to hypothesis. By Lemma 3, P/P' is a chief factor of G, and so by hypothesis has order p or p^2. Hence $o(p, q) = 1$ or 2. If $o(p, q) = 1$, $p \equiv 1(q)$ and G/P' is a nonabelian group of order pq, hence G/P' is isomorphic to $C_{p,q}$, a contradiction. If $o(p, q) = 2$, then $q > 2$ and $p \equiv -1(q)$. Since G/P' is in this case a nonabelian group of order p^2q it follows from Lemma 2 that if $p > 2$ then G/P' is isomorphic to $D_{p^2,q}$, again a contradiction. The only remaining possibility is that $p = 2$, $q = 3$ and $|G/P'| = 12$. But then since G/P' is nonabelian with a normal Sylow 2-subgroup, it follows that G is isomorphic to A^4, that is to $D_{2^2,3}$, a final contradiction.

Proof of Corollary 2. Again suppose the result false and let G be a counterexample of least possible order. Itô's theorem applies to show that G must be a minimal nonnilpotent group. But then G is soluble; and by hypothesis, every p-chief factor of G can be generated by 2 elements, hence $r_p(G) \leqslant 2$. Now Theorem A gives a contradiction.

To prove Theorem B′, we need a variant of a known result about extraspecial p-groups (see [7, p. 353, III.13.7]).

LEMMA 4. *Let P be a nonabelian p-group with $\Phi(P) = P' \leqslant Z(P)$ and $|P'| = p$. View $\hat{P} = P/P'$ as a vector space over* $\mathrm{GF}(p)$ *in the natural way: this vector space is a symplectic space relative to the bilinear form f defined (for all $x, y \in P$) by*

$$f(\hat{x}, \hat{y}) = a,$$

where $P' = \langle z \rangle$ and $[x, y] = z^a$ (and a is interpreted $(\mathrm{mod}\, p)$*). Then $\widehat{Z(P)}$ is the radical of $\hat{P}$ and if $|\widehat{Z(P)}| = p^r$ then $|P| = p^{r+2s+1}$ for some integer $s \geqslant 0$. Moreover, P has abelian subgroups of order p^{r+s+1} and elementary abelian subgroups of order p^{r+s}.*

Proof. For any particular choice of generator z of P', f is a well-defined bilinear form on $\hat{P}$, since $P' \leqslant Z(P)$, and f is obviously alternating. By definition, the radical of $\hat{P}$ is

$$\begin{aligned} &\{\hat{x} \in \hat{P} \mid f(\hat{x}, \hat{y}) = 0 \text{ for all } y \in P\} \\ &= \{\hat{x} \in \hat{P} \mid [x, y] = 1 \text{ for all } y \in P\} \\ &= \widehat{Z(P)}. \end{aligned}$$

We now apply the structure theorem for a symplectic space ([7, p. 217, II.9.6]). For some integer $s \geqslant 0$,

$$\hat{P} = \langle \widehat{Z(P)}, \hat{x}_1, \hat{y}_1, \ldots, \hat{x}_s, \hat{y}_s \rangle$$

where, for $i, j = 1, \ldots, s$,

$$f(\hat{x}_i, \hat{y}_i) = 1 \qquad \text{and whenever} \quad i \neq j$$
$$0 = f(\hat{x}_i, \hat{x}_j) = f(\hat{y}_i, \hat{y}_j) = f(\hat{x}_i, \hat{y}_j).$$

Thus $|P/Z(P)| = p^{2s}$, so that with $|Z(P)/P'| = p^r$,

$$|P| = p^{r+2s+1}.$$

The subgroup $A = Z(P)\langle x_1, \ldots, x_s \rangle$ has order p^{r+s+1} and is abelian. Then since $|A/\Phi(P)| = p^{r+s}$, A has an elementary abelian subgroup of order p^{r+s}.

Proof of Theorem B$'$. Suppose the result false and let G be a counterexample of least possible order. As before, G is a minimal nonnilpotent group with normal Sylow p-subgroup P and G/P is a cyclic q-group for some prime $q \neq p$. Again, $|G/P| = q$: otherwise G would have a normal q-subgroup $H \neq 1$ and then, since P would be isomorphic to PH/H, a Sylow p-subgroup of G/H, G/H would satisfy the same hypotheses as G. But then G/H would be p-nilpotent, hence also G would be p-nilpotent, contrary to hypothesis. By Lemma 3, P/P' is a chief factor of G, $|P/P'| = p^{o(p,q)}$ and P' is elementary abelian. Hence by hypothesis, $|P'| \leqslant p^2$.

If $P' = 1$ then also by hypothesis $|P| \leqslant p^2$ and so $o(p, q) = 1$ or 2. As in the proof of Theorem A′, this leads to the involvement in G of either $C_{p,q}$ or $D^*_{p^2,q}$, a contradiction.

Hence P is nonabelian and $|P'| = p$ or p^2. Let $P_0 < P'$ with $|P'/P_0| = p$. Since $P' \leqslant Z(G)$, P_0 is normal in G. We consider $\hat{G} = G/P_0$, with the "circumflex convention" applying. Then $\hat{P}' = \widehat{P'}$, of order p. In particular, $\hat{P}$ is nonabelian. Then, since $P' \leqslant Z(G)$, $\hat{P}' \leqslant Z(\hat{P}) < \hat{P}$. But $Z(\hat{P})$ is normal in $\hat{G}$ and $\hat{P}/\hat{P}'$ is a chief factor of $\hat{G}$. Therefore

$$\hat{P}' = Z(\hat{P}) = \Phi(\hat{P}):$$

thus $\hat{P}$ is extra-special. By Lemma 4,

$$|\hat{P}| = p^{2m+1}, \qquad \text{where} \quad 2m = o(p, q),$$

and $\hat{P}$ has abelian subgroups of order p^{m+1} and elementary abelian subgroups of order p^m.

Now we note that $m > 1$, and if $p = 2$, $m > 2$. For if $m = 1$ then $o(p, q) = 2$: this implies that $q > 2$ and $p \equiv -1(q)$ and, as in the proof of Theorem A′, it follows that G/P' is isomorphic to $D_{p^2,q}$, a contradiction. Also if $p = 2$ and $m = 2$ then $o(2, q) = 4$, which forces $q = 5$. But $\hat{G}$ is not 2-nilpotent: for if it were, so would G be, since $P_0 \leqslant Z(G)$. Hence $\hat{P}$ is an extra-special group of order 2^5 with an automorphism of order 5. It follows by Lemma 1 that $\hat{P}$ is isomorphic to T and hence that $\hat{G}$ is isomorphic to B^{32}, a contradiction.

Suppose that $|P'| = p$. Then $P_0 = 1$ and $\hat{G} = G$: so P has abelian subgroups of order p^{m+1} and elementary abelian subgroups of order p^m. Hence by hypothesis $m \leqslant 2$, which by what we have shown above implies that $p > 2$ and $m = 2$. But when $p > 2$, P has exponent p ([7, III.5.2c]) and so the abelian subgroups of P of order p^{m+1} are in this case elementary. Then by hypothesis $m + 1 \leqslant 2$, a contradiction.

Hence $|P'| = p^2$. Now for any $x \in P \backslash P'$, $\langle x \rangle P'$ is abelian, since $P' \leqslant Z(G)$. If $p > 2$, P has exponent p, in which case $\langle x \rangle P'$ is elementary abelian, hence by hypothesis has order $\leqslant p^2$. Thus if $p > 2$, $|P'| = p$, which we have seen to be impossible.

Therefore $p = 2$ and $|P'| = 2^2$. Let P_1/P_0 be an elementary abelian subgroup of $\hat{P}$ of largest possible order; we know that this is $\geqslant 2^m$. If P_1 were abelian, it would have an elementary subgroup of order 2^m and so by hypothesis $m \leqslant 2$, a possibility which has already been ruled out. Hence P_1 is nonabelian and

$$P_1' = P_0 = \Phi(P_1) \leqslant Z(P_1).$$

In fact $P_0 < Z(P_1)$: for we must have $P' \leqslant P_1$; if not we should have $\hat{P}_1 \cap \hat{P}' = 1$ with $\hat{P}' = Z(\hat{P})$, and then $\widehat{P_1 P'} = \hat{P}_1 \times \hat{P}'$, an elementary abelian subgroup of $\hat{P}$ of order $> |\hat{P}_1|$, contrary to the choice of P_1. Hence $P_0 < P' \leqslant P_1 \cap Z(P) \leqslant Z(P_1)$.

Let $|Z(P_1)/P_1'| = 2^r$, where $r > 0$. By Lemma 4,

$$|P_1| = 2^{r+2s+1}$$

from some integer $s > 0$ (since P_1 is nonabelian), and P_1 has elementary abelian subgroups of order 2^{r+s}. Hence

$$r + s \leqslant 2, \qquad r > 0, \qquad s > 0, \qquad r + 2s \geqslant m \qquad \text{and} \qquad m > 2.$$

These inequalities imply that

$$r = s = 1 \qquad \text{and} \qquad m = 3.$$

Hence $o(2, q) = 6$.

Then since $2^6 \equiv 1(q)$, it follows that $q = 3$ or 7. But $o(2, 3) = 2$ and $o(2, 7) = 3$: a final contradiction.

Proof of Corollary 4. Let α be an automorphism of P of prime order $q \neq p$, and consider the subgroup $G = \langle \alpha \rangle P$ of the holomorph of P. If G were p-nilpotent then $\langle \alpha \rangle$ would be a direct factor of G, and so α would act trivially on P, a contradiction. The result now follows from Theorem B.

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AUTOMORPHISM GROUPS OF GROUPS WITH TRIVIAL CENTRE

By JOHN S. ROSE

There is no systematic general procedure by which the automorphism group of a given group can be found. For a group G which possesses a proper characteristic subgroup H with trivial centralizer in G, Lemma 1.1 of the present paper gives a possible means of reducing the calculation of Aut G by identifying Aut G with a specific subgroup of Aut H. This method is applied in the rest of the paper to the determination of automorphism groups for various familiar classes of finite groups with trivial centre: semi-simple groups in the sense of H. Fitting ([4]) in §2, certain wreath products in §3, and relative holomorphs of abelian groups in §§4, 5. In particular, the results reveal various complete subgroups of groups already known to be complete: automorphism groups of direct products of non-abelian simple groups (see Burnside ([2] §71), Schenkman ([26] 95, Theorem III.4.t) and holomorphs of abelian groups of odd orders (G.A. Miller [14]). An example of interest which has perhaps not been recorded before is given in Corollary 5.6: the extended affine group of any finite field with more than 2 elements is a complete group. None of the complete groups produced by the methods to be discussed seems to have odd order, so that a question raised in ([24]) (and previously mentioned by G.A. Miller ([15])) is not answered here. †

Notation and terminology are mostly standard. For any group G, Aut G denotes the group of all automorphisms of G, Inn G the group of all inner automorphisms of G, and Out G = Aut G/Inn G. The holomorph of G, denoted by Hol G, is the semi-direct product of G by Aut G with natural action. When Z(G) = 1, G is identified with Inn G by identifying each element of G with the inner automorphism of G which it induces. A group G is called complete if Z(G) = 1 = Out G; equivalently (R. Baer [1]) if G is a direct factor of every

† I have learned recently that an example of a non-trivial complete group of odd order has been found by R.S. Dark, and will be described in an article to be published in Proc. Cambridge Philos. Soc.

group containing it as a normal subgroup. For any group A, a subgroup V of Aut A is said to be a fixed-point-free group of automorphisms of A (or to have fixed-point-free action on A) if $C_A(V) = 1$. (This adheres to the convention of Gorenstein ([7] 333) rather than that of Huppert ([11] 497, V.8.4)). If A is any abelian group other than an elementary 2-group, the corresponding generalized dihedral group Dih A is a subgroup of Hol A, namely the semi-direct product with natural action of A by the group of order 2 generated by the automorphism which inverts every element of A.

For any non-trivial finite group G, the product of all the minimal normal subgroups of G is denoted by S(G) and called the socle of G. It is known that S(G) is the direct product of some of the minimal normal subgroups of G (R. Remak [20]). The exponent of G is denoted by exp G. The symbol p is always used to denote a prime number, and π to denote a set of prime numbers; π' denotes the set of all primes not belonging to π. The unique largest normal π-subgroup of G is denoted by $O_\pi(G)$ and the unique smallest normal subgroup of G by which the quotient of G is a π-group is denoted by $O^\pi(G)$.

For any finite set X, the symmetric group on X is denoted by Σ_X. Since Σ_X is determined up to isomorphism by the cardinality of X, Σ_X is also often denoted by Σ_n, where $n = |X|$.

The greatest common divisor of integers m and n is denoted by (m, n). When (m, n) = 1 and n is positive, o(m, n) denotes the order of m (mod n), that is the least positive integer k such that $m^k \equiv 1$ (n).

For any group G and any positive integer n, nG denotes the direct product of n copies of G.

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1. Characteristic subgroups with trivial centralizer.

Suppose that G is a group which has a characteristic subgroup H such that $C_G(H) = 1$. Then $Z(G) = 1$ and we identify G with Inn G in the natural way. Restriction to H of automorphisms of G determines a homomorphism $\rho: \mathrm{Aut}\, G \to \mathrm{Aut}\, H$. Since $H \trianglelefteq G \trianglelefteq \mathrm{Aut}\, G$ and $C_G(H) = 1 = C_{\mathrm{Aut}\, G}(G)$, it follows by a result of H. Wielandt ([27] (45)) that $C_{\mathrm{Aut}\, G}(H) = 1$. This means that ρ is injective. Furthermore $G^\rho \leq \mathrm{Im}\, \rho \leq N_{\mathrm{Aut}\, H}(G^\rho)$.

We now define a map $\varphi: N_{\mathrm{Aut}\, H}(G^\rho) \to \mathrm{Im}\, \rho$ as follows. Each $x \in N_{\mathrm{Aut}\, H}(G^\rho)$ determines by conjugation of G^ρ in Aut H an automorphism of G^ρ which corresponds (via ρ) to a unique automorphism $\bar{x}$ of G. Let $x^\varphi = \bar{x}^\rho$. Then φ is evidently a homomorphism. It is injective because ρ is injective and $C_{\mathrm{Aut}\, H}(G^\rho) \leq C_{\mathrm{Aut}\, H}(H^\rho) = C_{\mathrm{Aut}\, H}(\mathrm{Inn}\, H) = 1$, since $Z(H) = 1$. Let $y \in \mathrm{Im}\, \rho$: then $y = \alpha^\rho$ for some $\alpha \in \mathrm{Aut}\, G$, and since, for all $g \in G$, $(\alpha^\rho)^{-1} g^\rho \alpha^\rho = (g^\alpha)^\rho$, it follows that $\bar{y} = \alpha$. Hence $y^\varphi = \bar{y}^\rho = y$. Therefore, for any $x \in N_{\mathrm{Aut}\, H}(G^\rho)$, $(x^\varphi)^\varphi = x^\varphi$, and so, since φ is injective, $x^\varphi = x$. Thus $\mathrm{Im}\, \rho = N_{\mathrm{Aut}\, H}(G^\rho)$.

We now use ρ to identify G and Aut G with subgroups of Aut H. This means that each element of G is identified with the automorphism of H which it induces by conjugation in G. This establishes

LEMMA 1.1. Let G be a group with a characteristic subgroup H such that $C_G(H) = 1$. Then G is naturally embedded in Aut H by means of conjugation of H by the elements of G, and there is a natural isomorphism between Aut G and $N_{\mathrm{Aut}\, H}(G)$.

A special case of this result is to be found in D.S. Passman's book ([18] 37, Theorem 5.9). Indeed, 1.1 can be reformulated in the following statement which is of the same kind as Passman's.

Let H be a group with $Z(H) = 1$ and let $H \leq G \leq \mathrm{Aut}\, H$. If H is characteristic in G then $\mathrm{Aut}\, G = N_{\mathrm{Aut}\, H}(G)$ (by a natural identification).

If G is a group and $H \trianglelefteq G$ but H is not characteristic in G then there is a normal subgroup H^* of G such that $H \neq H^* \cong H$. Then $H^*/H \cap H^* \cong HH^*/H \trianglelefteq G/H$. Hence G/H has a non-trivial normal subgroup which is isomorphic to a quotient of H^*, and hence also to a quotient of H. This remark gives

COROLLARY 1.2. Let H be a group with $Z(H) = 1$ and let $H \leq G \leq \mathrm{Aut}\, H$. If no non-trivial normal subgroup of G/H is isomorphic to a quotient of H then $\mathrm{Aut}\, G = N_{\mathrm{Aut}\, H}(G)$. In particular, $\mathrm{Aut}\, G = N_{\mathrm{Aut}\, H}(G)$ if any one of the following conditions holds:

i) H is perfect and G/H is soluble, or

ii) G is finite, H is a π-group and $O_{\pi}(G/H) = 1$, or

iii) G is finite, G/H is a π-group and $O^{\pi}(H) = H$.

It was shown in ([24], Corollary 3) that if H is any finite perfect group with $Z(H) = 1$ then there is a complete group G such that $H \leq G \leq \mathrm{Aut}\, H$ and G/H is soluble. Corollary 1.2 yields a different proof of this fact. Indeed 1.2 i) shows that if H is a finite perfect group with $Z(H) = 1$ and if G/H is any self-normalizing soluble subgroup of $\mathrm{Out}\, H$ then G is necessarily complete; and it is easy to see that any finite group Q must possess self-normalizing soluble subgroups, for any maximal soluble subgroup of Q must be self-normalizing in Q.

It was also shown in ([24] Corollary 1) that if J is a finite p'-group such that $Z(J) = 1$ and $\mathrm{Out}\, J$ is a p-group then $\mathrm{Aut}\, J$ is complete. This can now be generalized by ii) and iii) of 1.2 to give the following statements:

If H is a finite π-group such that $Z(H) = 1$ and $O_{\pi}(\mathrm{Out}\, H) = 1$ then $\mathrm{Aut}\, H$ is complete.

If H is a finite group such that $Z(H) = 1$, $O^{\pi}(H) = 1$ and $\mathrm{Out}\, H$ is a π-group then $\mathrm{Aut}\, H$ is complete.

In particular, in Lemma 5 of ([24]) $\mathrm{Aut}\, J$ is necessarily complete, even although in general $K \neq \mathrm{Aut}\, J$.

In principle Lemma 1.1 allows the determination of the automorphism group of a group G with trivial centre to be reduced to the determination of the automorphism group of a smaller and possibly more amenable group, providing that G has a proper characteristic subgroup with trivial centralizer in G.

The condition that $C_G(H) = 1$ for a subgroup H of G is equivalent to the condition that every subgroup of G containing H has trivial centre: for if $H \leq J \leq G$ and $C_G(H) = 1$ then obviously $Z(J) = 1$, and for the converse, if $H \leq G$ with $1 \neq x \in C_G(H)$ then $H \leq \langle H, x \rangle \leq G$ and $1 \neq x \in Z(\langle H, x \rangle)$.

When G is finite and $H \trianglelefteq G$ there is another equivalent condition which is easily verified:

LEMMA 1.3. Let H be a normal subgroup of a finite group G. Then $C_G(H) = 1$ if and only if $Z(H) = 1$ and $S(G) \leq H$.

If G is a finite group with $Z(G) = 1$ one might try to reduce the determination of Aut G as far as possible by applying 1.1 to a characteristic subgroup H of G which is minimal subject to $C_G(H) = 1$. By means of 1.3 and the result of Wielandt referred to above, it is easy to show that then $Z(H) = 1$ but $Z(K) \neq 1$ for every proper characteristic subgroup K of H which contains S(H). But of course the structure of H can be very complicated. Moreover in general H is not uniquely determined by G and the minimality condition. It is hoped to show nevertheless that 1.1 does provide a simple tool for the calculation of automorphism groups in several cases of interest.

For use in §§2, 3 we note the following simple consequence of the Krull-Remak-Schmidt theorem (Huppert [11] 70, I.12.6; see also Fitting [4] Satz 12).

LEMMA 1.4. Let G be a non-trivial finite group with trivial centre. Suppose that

$$G = n_1G^1 \times \dots \times n_sG^s,$$

where s, n_1, ..., n_s are positive integers and G^1, ..., G^s are pairwise non-isomorphic directly indecomposable groups. Then

$$\mathrm{Aut}\, G \cong (\mathrm{Aut}\, G^1) \wr \Sigma_{n_1} \times \ldots \times (\mathrm{Aut}\, G^s) \wr \Sigma_{n_s},$$

where the direct factors are natural wreath products.

2. Fitting's theory of semi-simple groups

In ([4]) H. Fitting defines a finite group G to be semi-simple if it has no abelian normal subgroup $\neq 1$, equivalently if $Z(S(G)) = 1$. (For an account of Fitting's theory, see (Kurosh [12] §61).)

Let G be a finite semi-simple group. Then by 1.3, S(G) is a characteristic subgroup of G with trivial centralizer in G, and is the unique minimal such subgroup. Now 1.1 applies to show that G is embedded in Aut S(G) (as proved by Fitting) and furthermore that Aut G is also embedded in Aut S(G) as $N_{\mathrm{Aut}\, S(G)}(G)$.

Here S(G) is a direct product of non-abelian simple groups. Now Fitting also showed (Kurosh [12] 207) that if S is any finite direct product of non-abelian simple groups and if G is any subgroup of Aut S which contains S then G is semi-simple and $S(G) = S$. Thus the extra information provided by 1.1 is expressed in

THEOREM 2.1. If S is any finite direct product of non-abelian simple groups and if $S \leq G \leq \mathrm{Aut}\, S$ then (by a natural identification) $\mathrm{Aut}\, G = N_{\mathrm{Aut}\, S}(G)$. In particular, if G is self-normalizing in Aut S then G is complete.

Now let s, n_1, ..., n_s be positive integers, E_1, ..., E_s pairwise non-isomorphic finite non-abelian simple groups, and

$$S = n_1E_1 \times \ldots \times n_sE_s.$$

By a natural identification we may by 1.4 set

$$\mathrm{Aut}\, S = (\mathrm{Aut}\, E_1) \wr \Sigma_{n_1} \times \ldots \times (\mathrm{Aut}\, E_s) \wr \Sigma_{n_s},$$

where the direct factors are natural wreath products. For $i = 1, \ldots, s$ let B_i denote the base group of the wreath product $(\mathrm{Aut}\, E_i) \wr \Sigma_{n_i}$: thus B_i is the direct product of n_i copies of $\mathrm{Aut}\, E_i$, and

$$\mathrm{Aut}\, S = \Sigma_{n_1} B_1 \times \ldots \times \Sigma_{n_s} B_s .$$

For an application of 2.1, consider a subgroup G of Aut S of the form $G = T_1 B_1 \times \ldots \times T_s B_s$, where, for each $i = 1, \ldots, s$, T_i is a self-normalizing subgroup of Σ_{n_i}. Then $S \leq G$ and by 2.1, $\mathrm{Aut}\, G = N_{\mathrm{Aut}\, S}(G)$. Let $D = B_1 \times \ldots \times B_s \trianglelefteq \mathrm{Aut}\, S$. Then $\mathrm{Aut}\, G/D = N_{\mathrm{Aut}\, S}(G)/D = N_{\mathrm{Aut}\, S/D}(G/D)$. But $\mathrm{Aut}\, S/D \cong \Sigma_{n_1} \times \ldots \times \Sigma_{n_s}$, by an isomorphism in which G/D corresponds to $T_1 \times \ldots \times T_s$. Hence, by hypothesis, G/D is self-normalizing in $\mathrm{Aut}\, S/D$. Therefore $\mathrm{Aut}\, G = G$, that is G is complete.

COROLLARY 2.2. Let $s, n_1, \ldots, n_s$ be positive integers and $E_1, \ldots, E_s$ pairwise non-isomorphic finite non-abelian simple groups. For each $i = 1, \ldots, s$ let T_i be a self-normalizing subgroup of Σ_{n_i}. Then the group $(\mathrm{Aut}\, E_1) \wr T_1 \times \ldots \times (\mathrm{Aut}\, E_s) \wr T_s$ is complete (where the wreath products are the natural ones determined by $T_1 \leq \Sigma_{n_1}, \ldots, T_s \leq \Sigma_{n_s}$).

3. Wreath products

The structure of the automorphism group of a regular wreath product W has been investigated by C.H. Houghton ([9]). Here we ask what information is provided by the application of Lemma 1.1 when W is finite and $Z(W) = 1$.

Let G and X be finite groups with $G \neq 1$ and let $W = G \wr X$, the regular wreath product of G by X. Let $D = \mathrm{Dr}\, G^X$, the base group of W. It is easy to verify that $C_W(D) \leq D$. Hence if $Z(G) = 1$ then $C_W(D) = 1$.

P.M. Neumann ([17] Theorem 9.12) proved that D is characteristic in W unless $G = \mathrm{Dih}\, A$ for some abelian group A of odd order and $|X| = 2$. Therefore, if $Z(G) = 1$ and either G is not generalized dihedral or $|X| \neq 2$ we can apply

1.1 and find that $\operatorname{Aut} W = N_{\operatorname{Aut} D}(W)$. We use this in conjunction with 1.4 to prove

THEOREM 3.1. Let G and X be finite groups and let W be the regular wreath product $G \wr X$. Suppose that $G \neq 1$, $Z(G) = 1$ and either $|X| \neq 2$ or G is not generalized dihedral. Let $G = n_1G^1 \times \dots \times n_sG^s$, where $s, n_1, \dots, n_s$ are positive integers and $G^1, \dots, G^s$ are pairwise non-isomorphic directly indecomposable groups. Let $D = \operatorname{Dr} G^X$, the base group of W. Then (by a natural identification) $\operatorname{Aut} W = N_{\operatorname{Aut} D}(W)$. Moreover, $\operatorname{Aut} W$ has a normal subgroup C and a subgroup H such that $\operatorname{Aut} W = CH$ and $C \cap H = D$, $C/D \cong n_1 \operatorname{Out} G^1 \times \dots \times n_s \operatorname{Out} G^s$, and there is a normal subgroup K of H and a subgroup J of H such that $H = KJ$, $K \cap J = D$, $K/D \cong (n_1 + \dots + n_s)X$ and $J/D \cong \Sigma_{n_1} \times \dots \times \Sigma_{n_s} \times \operatorname{Aut} X$. Here C/D and K/D centralize each other and $C/D \,.\, J/D \cong (\operatorname{Out} G^1) \wr \Sigma_{n_1} \times \dots \times (\operatorname{Out} G^s) \wr \Sigma_{n_s} \times \operatorname{Aut} X$, where the first s direct factors are natural wreath products. Also $D \leq W \leq H$, $W/D \cap K/D = Z(W/D)$, $W \cap J = D$ and C/D and W/D centralize each other.

Proof. For each $i = 1, \dots, s$ and for each ordered pair jx with $j \in \{1, \dots, n_i\}$ and $x \in X$ let G^i_{jx} be a copy of G^i in which, for each $g^i \in G^i$, g^i_{jx} is the corresponding element of G^i_{jx}. Then $D = \operatorname{Dr} \prod_{i,j,x} G^i_{jx}$ and in W, for each $y \in X$, $(g^i_{jx})^y = g^i_{jxy}$. By 1.4, $\operatorname{Aut} D = BT$, where $B = \operatorname{Dr} \prod_{i,j,x} (\operatorname{Aut} G^i)_{jx}$, $T = T_1 \times \dots \times T_s$, with T_i the symmetric group (of degree $n_i|X|$) on the set of all ordered pairs jx with $j \in \{1, \dots, n_i\}$ and $x \in X$, and where for each $\alpha^i \in \operatorname{Aut} G^i$ and $\tau = (\sigma^1, \dots, \sigma^s) \in T$ (with $\sigma^i \in T_i$),

$$(\alpha^i_{jx})^\tau = \alpha^i_{(jx)\sigma^i} .$$

Now $W = DX$ is identified with a subgroup of $\operatorname{Aut} D$, with D identified as a subgroup of B in the obvious way and X identified with a subgroup of T by identification of $y \in X$ with the element

$$\left(\begin{pmatrix} jx \\ jxy \end{pmatrix}, \dots, \begin{pmatrix} jx \\ jxy \end{pmatrix} \right) \text{ of } T.$$

We know that $\mathrm{Aut}\,W/D = N_{\mathrm{Aut}\,D}(W)/D = N_{\mathrm{Out}\,D}(W/D)$, and by a natural identification $\mathrm{Out}\,D = \overline{B}T$, where $\overline{B} = B/D = \mathrm{Dr} \prod_{i,j,x} (\mathrm{Out}\,G^i)_{jx}$. In Out D, W/D is identified with $X \le T$. One can check readily (or apply 3.1 of ([22]) to show) that $N_{\mathrm{Out}\,D}(X) = C_{\overline{B}}(X)N_T(X)$, with $C_{\overline{B}}(X) \trianglelefteq N_{\mathrm{Out}\,D}(X)$. Clearly $C_{\overline{B}}(X) = \mathrm{Dr} \prod_{i,j} \{ \prod_{x \in X} \bar{\alpha}^i_{jx} : \bar{\alpha}^i \in \mathrm{Out}\,G^i\} \cong n_1 \mathrm{Out}\,G^1 \times \dots \times n_s \mathrm{Out}\,G^s$.

Let $\tau = (\sigma^1, \dots, \sigma^s) \in T$ and $y = \left(\begin{pmatrix} jx \\ jxy \end{pmatrix}, \dots, \begin{pmatrix} jx \\ jxy \end{pmatrix}\right) \in X$. Then $y^\tau = \left(\begin{pmatrix} (jx)\sigma^1 \\ (jxy)\sigma^1 \end{pmatrix}, \dots, \begin{pmatrix} (jx)\sigma^s \\ (jxy)\sigma^s \end{pmatrix}\right)$. If $\tau \in N_T(X)$ then the map $y \mapsto y^\tau$ is an automorphism, μ say, of X, and for $i = 1, \dots, s$,

$$\begin{pmatrix} (jx)\sigma^i \\ (jxy)\sigma^i \end{pmatrix} = \begin{pmatrix} jx \\ jx(y\mu) \end{pmatrix}.$$

We may define a map π_i of the set $\{1, \dots, n_i\}$ into itself and elements $u^i_j \in X$ for $j = 1, \dots, n_i$ by

$$(j1)\sigma^i = (j\pi_i)u^i_j,$$

and then by the preceding equation, for all $x \in X$

$$(jx)\sigma^i = (j\pi_i)u^i_j(x\mu).$$

It follows from this that $\pi_i \in \Sigma_{n_i}$ and

$$\sigma^i = \begin{pmatrix} jx \\ (j\pi_i)u^i_j(x\mu) \end{pmatrix}.$$

Conversely, if μ is any automorphism of X and, for each $i = 1, \dots, s$, π_i is any element of Σ_{n_i} and $u^i_{n_i}, \dots, u^i_{n_i}$ are any n_i elements of X, then

$$\sigma^i = \begin{pmatrix} jx \\ (j\pi_i)u^i_j(x\mu) \end{pmatrix} \qquad (*)$$

is an element of T_i; and $\tau = (\sigma^1, \ldots, \sigma^s)$ is an element of $N_T(X)$. Let the element σ^i of T_i defined by (*) be denoted by $(\pi_i, \mu; (u_1^i)^{-1}, \ldots, (u_{n_i}^i)^{-1})$. Then one checks that the rule of composition of T_i-components of elements of $N_T(X)$ is

$$(\pi_i,\mu;\, x_1^i, \ldots, x_{n_i}^i)(\rho_i,\nu;\, y_1^i, \ldots, y_{n_i}^i) = (\pi_i\rho_i,\mu\nu;\, x_1^i\nu y^i_{1\pi_i}, \ldots, x^i_{n_i}\nu y^i_{n_i\pi_i}).$$

If $\sigma^i = (\pi_i, \mu; x_1^i, \ldots, x_{n_i}^i)$ let $\sigma_o^i = (\pi_i, \mu; 1, \ldots, 1)$. Then the map $\psi : (\sigma^1, \ldots, \sigma^s) \mapsto (\sigma_o^1, \ldots, \sigma_o^s)$ is an endomorphism of $N_T(X)$ with $\mathrm{Im}\,\psi \cong \Sigma_{n_1} \times \ldots \times \Sigma_{n_s} \times \mathrm{Aut}\,X$ and $\mathrm{Ker}\,\psi \cong (n_1 + \ldots + n_s)X$. Moreover, $N_T(X)$ obviously splits over $\mathrm{Ker}\,\psi$. We note also that $y \in X$ has T_i-component $(1, \tilde{y}; y^{-1}, \ldots, y^{-1})$, where $\tilde{y}$ is the inner automorphism of X induced by y. Therefore $X \cap \mathrm{Ker}\,\psi = Z(X)$ and $X \cap \mathrm{Im}\,\psi = 1$.

The action of $N_T(X)$ on $C_{\overline{B}}(X)$ is described as follows. Let $\alpha^i \in \mathrm{Out}\,G^i$, $j \in \mathrm{Out}\,G^i$, $j \in \{1, \ldots, n_i\}$ and $\tau \in N_T(X)$, where τ has T_i-component $(\pi_i, \mu; x_1^i, \ldots, x_{n_i}^i)$. Then

$$\Big(\prod_{x \in X} \bar{\alpha}^i_{jx}\Big)^\tau = \prod_{x \in X} \bar{\alpha}^i_{(j\pi_i)(x_j^i)^{-1}(x\mu)} = \prod_{x \in X} \bar{\alpha}^i_{(j\pi_i)x}.$$

Hence $\mathrm{Ker}\,\psi$ centralizes $C_{\overline{B}}(X)$ and

$$C_{\overline{B}}(X) \,.\, \mathrm{Im}\,\psi \cong (\mathrm{Out}\,G^1) \wr \Sigma_{n_1} \times \ldots \times (\mathrm{Out}\,G^s) \wr \Sigma_{n_s} \times \mathrm{Aut}\,X.$$

This completes the proof.

We shall now suppose in 3.1 that all extensions of G split, equivalently ([23] Corollary 2.3) that $Z(G) = 1$ and $\mathrm{Aut}\,G$ splits over G. We note some conditions under which all extensions of W split.

COROLLARY 3.2. Let the notation and hypotheses be as in 3.1. Suppose further that all extensions of G split. Then all extensions of W split if any one of the following extra conditions holds:

(a) G is indecomposable (that is, $s = 1 = n_1$), or

(b) all extensions of X split, or

(c) X is abelian and $(n_1 + \dots + n_s, |X|) = 1$.

But there is an extension of W which does not split if $s = 1$, $n_1 = p$, prime, and $|X| = p$.

Proof. The class of finite groups all extensions of which split is closed under the formation of finite direct products ([25] Theorem 2). Hence all extensions of D split.

Since $Z(W) = 1$, all extensions of W split if and only if $\mathrm{Aut}\, W$ splits over W; by the preceding remark, that is if and only if $\mathrm{Aut}\, W/D$ splits over W/D, hence by 3.1 if and only if H/D splits over W/D. Thus, with the notation of the proof of 3.1, all extensions of W split if and only if $N_T(X)$ splits over X. The assertion under (b) follows immediately.

If G is indecomposable then $T = \Sigma_X$, X is embedded in T by means of the regular representation, and $N_T(X) = \mathrm{Hol}\, X$. In this case $\mathrm{Aut}\, X$ is a complement to X in $N_T(X)$, and the assertion under (a) follows.

Now suppose that X is abelian: then by 3.1, $W \leq K$. Let $\overline{K} = \mathrm{Ker}\, \psi$ and $\overline{J} = \mathrm{Im}\, \psi$ (where ψ is as in the proof of 3.1), the subgroups of $N_T(X)$ corresponding in the natural isomorphism between H/D and $N_T(X)$ to K/D and J/D, respectively. Then X corresponds to W/D and $X \leq \overline{K}$. If $N_T(X)$ splits over X, let U be a complement to X in $N_T(X)$: then $U \cap \overline{K}$ is a complement to X in $\overline{K}$ and since $\overline{K}$ is abelian, $U \cap \overline{K} \trianglelefteq U\overline{K} = N_T(X)$. If, conversely, V is a complement to X in $\overline{K}$ and $V \trianglelefteq N_T(X)$ then $\overline{J}V$ is a complement to X in $N_T(X)$. Hence in this case all extensions of W split if and only if X is complemented in $\overline{K}$ by a normal subgroup of $N_T(X)$.

Each element τ of $N_T(X)$ determines $\pi_1 \in \Sigma_{n_1}, \dots, \pi_s \in \Sigma_{n_s}$, $\mu \in \mathrm{Aut}\, X$

and $n_1 + \dots + n_s$ elements x_j^i of X ($j = 1, \dots, n_i$; $i = 1, \dots, s$) such that $\tau = (\sigma^1, \dots, \sigma^s)$ and, for $i = 1, \dots, s$, $\sigma^i = (\pi_i, \mu; x_1^i, \dots, x_{n_i}^i)$, in the notation of the proof of 3.1. Here $\tau \in \overline{K}$ if and only if $\pi_1 = 1, \dots, \pi_s = 1$, $\mu = 1$, whereas $\tau \in \overline{J}$ if and only if $x_j^i = 1$ for all i, j. The action of $\overline{J}$ on $\overline{K}$ is determined by the following equation for T_i-components of elements:

$$(1, 1; x_1^i, \dots, x_{n_i}^i)^{(\pi_i, \mu; 1, \dots, 1)} = (1, 1; x^i_{1\pi_i^{-1}}\mu, \dots, x^i_{n_i\pi_i^{-1}}\mu).$$

Let V be the subgroup of the abelian group $\overline{K}$ consisting of all elements τ as above such that $\prod_{i=1}^{s} \prod_{j=1}^{n_i} x_j^i = 1$. Clearly $V \trianglelefteq \overline{J}\,\overline{K} = N_T(X)$. Now as a subgroup of $\overline{K}$, X consists of those elements τ for which x_j^i has a constant value for all i, j. Therefore $V \cap X = 1$ if $(n_1 + \dots + n_s, |X|) = 1$, in which case V is a complement to X in $\overline{K}$, since $|V| = |\overline{K}/X|$. This establishes the assertion under (c).

Finally, suppose that $s = 1$ and $n_1 = p = |X|$. Then $\overline{K} \cong pX$ and $\overline{J} \cong \Sigma_p \times \operatorname{Aut} X$, where Aut X is cyclic of order $p - 1$. Let **P** be a Sylow p-subgroup of $\overline{J}$: then $|P| = p$ and $P\overline{K} \cong$ the regular wreath product $X \wr X$. Then $|Z(P\overline{K})| = p$. Therefore, since $X \trianglelefteq P\overline{K}$ and $|X| = p$, $X = Z(P\overline{K})$. Now every non-trivial normal subgroup of $N_T(X)$ contained in $\overline{K}$ is normal in $P\overline{K}$ and so contains X. Hence X is not complemented in $\overline{K}$ by a normal subgroup of $N_T(X)$, and so the extensions of W do not all split in this case.

Remark. 3.2 would fail if G were allowed to be generalized dihedral and $|X| = 2$. One can show for instance that if W is the wreath product of Σ_3 by a group of order 2 then Aut W does not split over W.

P.M. Neumann ([17] 373) observed that if G is an indecomposable complete group, not isomorphic to Σ_3, then the wreath product of G by a group of order 2 is complete. We note

COROLLARY 3.3. Let G and X be non-trivial finite groups and let

$W = G \wr X$, the regular wreath product. Then W is complete if and only if G is complete, indecomposable and not isomorphic to Σ_3, and $|X| = 2$.

C.H. Houghton ([9]) discussed some instances of wreath products with soluble automorphism groups. We note also

COROLLARY 3.4. Let G and X be finite groups with $Z(G) = 1$ and let W be the regular wreath product $G \wr X$. If $\mathrm{Aut}\, G$ and $\mathrm{Aut}\, X$ are both soluble then $\mathrm{Aut}\, W$ is soluble.

In order to prove 3.3 and 3.4 we need some information about the exceptional cases excluded from 3.1.

LEMMA 3.5. Let A be an abelian group of odd order > 1, $G = \mathrm{Dih}\, A$, X a group of order 2, and W the regular wreath product $G \wr X$. Then $\mathrm{Aut}\, W$ has a series $\mathrm{Aut}\, W > L > M > 1$ with $\mathrm{Aut}\, W/L$ dihedral of order 8, $L/M \cong \mathrm{Aut}\, A$ and $M \cong A \times A$.

Proof. $W = (A_1\langle t_1\rangle \times A_2\langle t_2\rangle)\langle x\rangle$, where A_1 and A_2 are copies of A, with each $a \in A$ corresponding to elements $a_1 \in A_1$ and $a_2 \in A_2$ respectively, $t_1^2 = t_2^2 = x^2 = 1$, $t_1^x = t_2$, $a_1^x = a_2$ for all $a \in A$, and $a_i^{t_i} = a_i^{-1}$ for $i = 1, 2$. Let $Y = A_1 \times A_2$, a characteristic subgroup of W of odd order with W/Y dihedral of order 8. Then $\mathrm{Aut}\,(W/Y)$ is also dihedral of order 8, and the natural homomorphism $\psi : \mathrm{Aut}\, W \to \mathrm{Aut}\,(W/Y)$ is in fact an epimorphism: for $\mathrm{Aut}\,(W/Y)$ is generated by the images under ψ of the inner automorphism of W induced by x and the automorphism α of W such that $a_1^\alpha = a_1^{-1}a_2$ for all $a \in A$, $t_1^\alpha = x$ and $x^\alpha = t_1$.

Let $L = \mathrm{Ker}\,\psi$. If $\gamma \in L$ and $a \in A$ with, say, $a_1^\gamma = b_1c_2$, where $b, c \in A$, then, since $a_1^{t_1} = a_1^{-1}$ and $t_1^\gamma \equiv t_1 \pmod{Y}$, $(b_1c_2)^{t_1} = (b_1c_2)^{-1}$. Therefore, since $|A|$ is odd, $c = 1$. Hence γ leaves A_1 invariant, so that γ determines by restriction to A_1 an automorphism, $\hat{\gamma}$ say, of A such that

$$a_1^\gamma = (a^{\hat{\gamma}})_1 \text{ for all } a \in A.$$

Moreover, since $a_1^x = a_2$ and $x^\gamma \equiv x(\mathrm{mod}\, Y)$, γ leaves A_2 invariant and

$$a_2^\gamma = (a^{\hat{\gamma}})_2 \text{ for all } a \in A.$$

Now the map $\chi : \gamma \mapsto \hat{\gamma}$ is obviously an epimorphism $L \to \mathrm{Aut}\, A$.

Let $M = \mathrm{Ker}\, \chi$, the stability group of W with respect to Y. Let $\gamma \in M$. Then, since $\gamma \in L$, $t_1^\gamma = b_1 c_2 t_1$ with b, $c \in A$. Then $1 = (t_1^\gamma)^2 = c_2^2$, so that $c = 1$. Also $x^\gamma = d_1 e_2 x$ with d, $e \in A$. Then $1 = (x^\gamma)^2 = d_1 e_1 d_2 e_2$, so that $e = d^{-1}$. Now the map $\Theta : M \to A \times A$ defined by

$$\Theta : \gamma \mapsto (b, d),$$

where $t_1^\gamma = b_1 t_1$ and $x^\gamma = d_1 d_2^{-1} x$, is a homomorphism. It is an isomorphism since γ is uniquely determined by the last two equations, and for any $(b, d) \in A \times A$ there is an automorphism γ of W such that

$$a_1^\gamma = a_1 \text{ for all } a \in A,\ t_1^\gamma = b_1 t_1 \text{ and } x^\gamma = d_1 d_2^{-1} x.$$

Proof of 3.3. $Z(W) = 1$ if and only if $Z(G) = 1$. Assume that $Z(G) = 1$, and $G = n_1 G^1 \times \dots \times n_s G^s$, in the notation of 3.1. Suppose first that either $|X| \neq 2$ or G is not generalized dihedral. Then W is complete if and only if $\mathrm{Aut}\, W = W$, which by 3.1 is true if and only if $C/D \,.\, H/D = X$, hence if and only if $s = 1 = n_1$ and $\mathrm{Out}\, G = 1 = \mathrm{Aut}\, X$, that is G is complete and indecomposable and $|X| = 2$. Now suppose that $|X| = 2$ and G is generalized dihedral. Since $Z(G) = 1$, $G = \mathrm{Dih}\, A$, where A is abelian of odd order > 1. If W were complete we should have

$$8|A|^2 = |W| = |\mathrm{Aut}\, W| = 8|\mathrm{Aut}\, A|\ |A|^2, \text{ by 3.5,}$$

hence $\mathrm{Aut}\, A = 1$, which is impossible for $|A|$ odd > 1. Moreover, for A abelian of odd order > 1, Dih A is complete if and only if $\mathrm{Dih}\, A = \mathrm{Hol}\, A$ (Miller, Blichfeldt, Dickson [16] 169), that is if and only if $|\mathrm{Aut}\, A| = 2$, hence if and only if $\mathrm{Dih}\, A \cong \Sigma_3$.

Proof of 3.4. Suppose that Aut G and Aut X are both soluble. We may assume that $G \neq 1 \neq X$. Assume first that either $|X| \neq 2$ or G is not generalized dihedral, and let $G = n_1 G^1 \times \dots \times n_s G^s$, in the notation of 3.1. Since by 1.4, $\mathrm{Aut} \cong (\mathrm{Aut}\, G^1) \wr \Sigma_{n_1} \times \dots \times (\mathrm{Aut}\, G^s) \wr \Sigma_{n_s}$, it follows by hypothesis that for $i = 1, \dots, s$, $\mathrm{Aut}\, G^i$ is soluble and $n_i \leq 4$. Now in 3.1, D, C/D, K/D and J/D are all soluble. Hence Aut W is soluble. Now assume that $|X| = 2$ and G is generalized dihedral. Since $Z(G) = 1$, $G = \mathrm{Dih}\, A$ for some abelian group A of odd order > 1. Since $\mathrm{Aut}\, G = \mathrm{Hol}\, A$ (Miller, Blichfeldt, Dickson, loc cit.) and Aut G is soluble, it follows that Aut A is soluble. Hence by 3.5, Aut W is soluble.

4. Faithful fixed-point-free actions on finite abelian groups

From now on A always stands for a finite abelian group $\neq 1$. A subgroup of Hol A which contains A is called a relative holomorph of A. Various results are known about automorphism groups of relative holomorphs: for instance, $\mathrm{Aut}(\mathrm{Dih}\, A) \cong \mathrm{Hol}\, A$ (Miller, Blichfeldt, Dickson [16] 169) and Hol A itself is complete if $|A|$ is odd (Miller [14]). Following earlier work of Yu. A. Gol'fand ([6]), Nai-Chao Hsu ([10]) has investigated the groups of automorphisms of relative holomorphs of A which leave A invariant. (See also Plotkin [19] §6.3). Here we shall show that under various conditions, if G is a relative holomorph of A with $Z(G) = 1$ then A is characteristic in G and Aut G can be identified with a specific subgroup of Hol A.

Let $A \leq G \leq \mathrm{Hol}\, A$. Then $G = VA$, where $V = G \cap \mathrm{Aut}\, A$, and $C_{\mathrm{Hol}\, A}(G) = C_{\mathrm{Hol}\, A}(V) \cap C_{\mathrm{Hol}\, A}(A) = C_A(V)$. Hence also $Z(G) = C_A(V)$. Thus $Z(G) = 1$ if and only if V is a fixed-point-free group of automorphisms of A, and if so then $C_{\mathrm{Hol}\, A}(G) = 1$.

Hence if $Z(G) = 1$ then $N_{\mathrm{Hol}\, A}(G)$ can be embedded in Aut G. Note also that since

$$N_{\mathrm{Hol}\,A}(G)/A = N_{\mathrm{Hol}\,A\,/A}(VA/A) \cong N_{\mathrm{Aut}\,A}(V) \cong N_{\mathrm{Aut}\,A}(V)A/A \le N_{\mathrm{Hol}\,A}(G)/A,$$

$N_{\mathrm{Hol}\,A}(G) = N_{\mathrm{Aut}\,A}(V)A$. We shall show that under various conditions, Aut G can be identified with $N_{\mathrm{Aut}\,A}(V)A$.

Before bringing Lemma 1.1 into play we deal directly with two special cases in the following theorem.

THEOREM 4.1. Suppose that V is a fixed-point-free group of automorphisms of A and let $G = VA \le \mathrm{Hol}\,A$. Suppose further that either (a) $(|V|, |A|) = 1$ or (b) V is nilpotent. Then $[V, A] = A$ and, by a natural identification, $\mathrm{Aut}\,G = N_{\mathrm{Aut}\,A}(V)A$.

Proof. (i) We prove first that $[V, A] = A$. If $(|V|, |A|) = 1$ then, by a theorem of Zassenhaus (Huppert [11] 350, III.13.4)

$$A = [V, A] \times C_A(V) = [V, A], \text{ as required.}$$

Now suppose that V is nilpotent; in this case we argue by induction on $|A|$. If $A = A_1 \times A_2$, where A_1 and A_2 are V-invariant proper subgroups of A then, by the induction hypothesis $[V, A_i] = A_i$, for i = 1, 2, and so $[V, A] = A$. Therefore, we may assume that A is indecomposable as a V-module, and in particular that A is a p-group for some p. Now we may write $V = P \times R$, where P is a p-group and R is a nilpotent p'-group. Consider the action of P on the V-invariant subgroup $B = C_A(R)$ of A. Since $C_B(P) = B \cap C_A(P) = C_A(V) = 1$ and P and B are both p-groups, we must have B = 1. Thus the action of R on A is fixed-point-free. Since $(|R|, |A|) = 1$, it follows from case (a) that $[R, A] = A$. Hence also $[V, A] = A$. This completes the induction argument.

(ii) Next, we show that A is characteristic in G. In case (a) this is clear, for then A is a normal Hall subgroup of G. In case (b), we show by induction on m that for any positive integer m, $K_m(G) = K_m(V)A$, where $K_m(G)$

denotes the m-th term of the lower central series of G (and similarly for V). This is trivial for $m = 1$, so assume that $m > 1$ and inductively that $K_{m-1}(G) = K_{m-1}(V)A$. Then $K_m(G) = [K_{m-1}(G), G] = [K_{m-1}(V), V][K_{m-1}(V), A][A, V]$, since A is abelian. Hence by (i) $K_m(G) = K_m(V)A$. This completes the induction argument. Now if V is nilpotent of class c, $K_{c+1}(G) = A$ and so A is characteristic in G.

(iii) Now we observe that $H^1(V, A) = 0$, where A is viewed in the natural way as a V-module. In case (a) this is immediate (MacLane [13], 117, Proposition 5.3). In case (b) we note that since $C_A(V) = 1$, $H^0(V, A) = 0$, a fortiori the Tate cohomology group $\hat{H}^0(V, A) = 0$. Then the assertion follows from a result of Hoechsmann, Roquette and Zassenhaus ([8]) and Wong ([28]).

(iv) The completion of the proof follows closely the proof of Lemma 3.3 of ([23]). By (ii), each automorphism θ of G determines by restriction an automorphism θ_1 of A. In G, $\eta^{-1}b\eta = b^{\eta}$ for all elements b in A and η in V. Application of θ to this equation shows (as in [23]) that $\theta_1 \in N_{\mathrm{Aut}\ A}(V) = N$, say. The map $\theta \mapsto \theta_1$ therefore defines a homomorphism $\chi\colon \mathrm{Aut}\ G \to N$. This is in fact an epimorphism, for if $\alpha \in N$ then the map $\alpha^*\colon G \to G$, defined by $\alpha^*\colon \eta b \mapsto (\alpha^{-1}\eta\alpha)b^{\alpha}$ for all elements b in A and η in V, is easily shown to be an automorphism of G such that $\alpha^*\chi = \alpha$.

Let $K = \mathrm{Ker}\,\chi$. If $\theta \in K$ then, as in ([23]) θ fixes both A and G/A pointwise. Hence (MacLane [13] 106, Proposition 2.1) K is isomorphic to the group of all crossed homomorphisms of V to the V-module A. It follows from (iii) that every crossed homomorphism of V to A is principal. Therefore (MacLane, loc. cit.) K consists of the inner automorphisms of G induced by elements of A.

For each element $a \in A$ let τ_a denote the inner automorphism of G induced by a. Then $K = \{\tau_a : a \in A\} \cong A$, since $Z(G) = 1$. Now there is a short exact sequence $1 \to K \to \mathrm{Aut}\ G \xrightarrow{\chi} N \to 1$. Moreover, this sequence splits: for the map $\xi: N \to \mathrm{Aut}\,G$ defined by $\xi: \alpha \mapsto \alpha^*$ for all $\alpha \in N$ is a homomorphism such that $\xi\chi$ = identity on N. Hence $\mathrm{Aut}\,G = N^*K$, where $N^* = N\xi$, $K \trianglelefteq \mathrm{Aut}\,G$ and $N^* \cap K = 1$. Now each element of Aut G is uniquely expressible in the form $\alpha^*\tau_a$ with $\alpha \in N$ and $a \in A$. Furthermore, it is easy to check that the map $\alpha^*\tau_a \mapsto \alpha a$ is a monomorphism $\mathrm{Aut}\,G \to \mathrm{Hol}\,A$.

We use this monomorphism to identify Aut G with a subgroup of Hol A. Then Aut G = NA, as asserted.

Remark. It can happen that the group G in 4.1 is a complete proper subgroup of Hol A. In 4.1, G is complete if and only if V is self-normalizing in Aut G, in particular only if $V \geq Z(\mathrm{Aut}\,A)$. It is known that $|Z(\mathrm{Aut}\,A)| = \varphi(m)$, where $m = \exp A > 1$ and φ is Euler's function (Miller, Blichfeldt, Dickson [16] 101). If, say, $m = \prod_{i=1}^{s} p_i^{t_i}$, where $p_1, \ldots, p_s$ are distinct primes and $s, t_1, \ldots, t_s$ positive integers, then $\varphi(m) = \prod_{i=1}^{s} p_i^{t_i-1}(p_i-1)$ and this has to divide $|V|$. In case (a) of 4.1, $(|V|, m) = 1$, so that then we must have $t_i = 1$ for all i and G can be complete only if A is a direct product of elementary groups. An instance in which G is complete and $G \neq \mathrm{Hol}\,A$ occurs when A is elementary of order 3^2 and V is a Sylow 2-subgroup of Aut A.

By means of 4.1 (b) we can now prove

THEOREM 4.2.† Let G be a finite group with an abelian normal subgroup A such that G/A is nilpotent and Z(G) = 1. Then G splits over A, and there is a nilpotent subgroup V of Aut A such that $G \cong$ the subgroup VA of Hol A and $\mathrm{Aut}\,G \cong$ the subgroup $N_{\mathrm{Aut}\,A}(V)A$ of Hol A.

Proof. G acts on A by conjugation. Since A is abelian, this action naturally induces an action of G/A on A which, since Z(G) = 1, is fixed-point-free. Hence $\hat{H}^0(G/A, A) = 0$ and so, since G/A is nilpotent, it follows by the theorem of Hoechsmann, Roquette and Zassenhaus ([8]) and Wong ([28]) used in the proof of 4.1 that $H^2(G/A, A) = 0$. Therefore (MacLane [13] 112, Theorem 4.1) G splits over A.

† I am grateful to Professor D. J. S. Robinson for pointing out that a closely related result is contained in a paper of E. Schenkman, 'The splitting of certain solvable groups', Proc. Amer. Math. Soc. 6 (1955) 286-90.

Let V be a complement to A in G. Then $C_V(A) \trianglelefteq V$. If $C_V(A) \neq 1$ then, since V is nilpotent, $C_V(A) \cap Z(V) \neq 1$: but this would imply that $Z(G) \neq 1$, contrary to hypothesis. Hence $C_V(A) = 1$, that is the action of V on A is faithful. Thus we may assume that $V \leq \mathrm{Aut}\, A$ and $G = VA \leq \mathrm{Hol}\, A$. Now the result follows from 4.1.

COROLLARY 4.3. Let G be a finite group with a cyclic normal subgroup A such that G/A is nilpotent and $Z(G) = 1$. Then (i) G/A is abelian, (ii) $A = G'$, (iii) $|A|$ is odd, (iv) $\mathrm{Aut}\, G \cong \mathrm{Hol}\, A$, and (v) Aut G is complete.

Proof. By 4.2, we may assume that $G = VA \leq \mathrm{Hol}\, A$, where V is a nilpotent subgroup of Aut A, and $\mathrm{Aut}\, G \cong N_{\mathrm{Aut}\, A}(V)A \leq \mathrm{Hol}\, A$. Since A is cyclic, Aut A is abelian, and so, since $G/A \cong V$, G/A is abelian. Also $N_{\mathrm{Aut}\, A}(V) = \mathrm{Aut}\, A$, so that $\mathrm{Aut}\, G \cong \mathrm{Hol}\, A$. This proves (i) and (iv).

Since $Z(G) = 1$, the action of V on A is fixed-point-free, and so by 4.1, $A = [V, A] \leq G'$. Since also G/A is abelian, it follows that $A = G'$. Since A is cyclic, an element of order 2 in A would lie in Z(G). Therefore, since $Z(G) = 1$, $|A|$ must be odd. This proves (ii) and (iii). Now (v) follows from (iii) and (iv) and the fact that the holomorph of a cyclic group of odd order is complete (see, for instance, Burnside([2])).

4.3 (v) contains results of L. Gerhards ([5] Folgerungen 4.9.1, 4.9.2).

We call a group cyclic-by-nilpotent if it has a cyclic normal subgroup with nilpotent quotient group; metacyclic if it has a cyclic normal subgroup with cyclic quotient group.

COROLLARY 4.4. (i) The only finite complete cyclic-by-nilpotent groups are the holomorphs of cyclic groups of odd orders.

(ii) The only finite complete metacyclic groups are the holomorphs of cyclic groups of odd prime power orders.

Proof. The holomorph of a cyclic group G of odd order is known to be complete, and it is clearly cyclic-by-nilpotent, metacyclic if $|G|$ is a prime power. Conversely, let G be a finite complete cyclic-by-nilpotent group. Then by 4.3, $G \cong \mathrm{Aut}\, G \cong \mathrm{Hol}\, G'$ and G' is cyclic of odd order. This proves (i). Now suppose that G is a finite complete metacyclic group. Then, as above, $G \cong \mathrm{Hol}\, G'$ and so G and Hol G' have derived groups of the same order. Also $\mathrm{Hol}\, G'/G' \cong \mathrm{Aut}\, G'$, which is abelian because G' is cyclic. Thus G' is the derived group of both G and Hol G'. But by 4.3 (ii), G/G' is cyclic, and therefore Aut G' is also cyclic. Since $|G'|$ is odd, this implies that $|G'|$ must be an odd prime power, and completes the proof of (ii).

Next we note a condition for the action on A of a subgroup V of Aut A to be fixed-point-free in terms of the actions on the Sylow subgroups of A.

LEMMA 4.5. Let the distinct prime divisors of $|A|$ be $p_1, \ldots, p_s$ with say, $|A| = \prod_{i=1}^{s} p_i^{n_i}$. For $i = 1, \ldots, s$ let A_i denote the Sylow p_i-subgroup of A and identify Aut A with $\mathrm{Aut}\, A_1 \times \ldots \times \mathrm{Aut}\, A_s$ in the natural way. Let ρ_i denote the projection map $\mathrm{Aut}\, A \to \mathrm{Aut}\, A_i$ and, for any subgroup V of Aut A, let $V_i = V^{\rho_i}$, so that V is a subdirect product of $V_1 \times \ldots \times V_s$. Then the action of V on A is fixed-point-free if and only if for every $i = 1, \ldots, s$, the action of V_i on A_i is fixed-point-free (hence if and only if the action of $V_1 \times \ldots \times V_s$ on A is fixed-point-free).

Proof. If, for some j, V_j fixes $a_j \in A_j$ with $a_j \neq 1$ then, since V_i fixed every point of A_j whenever $i \neq j$, it follows that $V_1 \times \ldots \times V_s$ fixes a_j.

Conversely, suppose that $a_1 \ldots a_s$ is a point $\neq 1$ of A which is fixed by V, where $a_i \in A_i$ for all $i = 1, \ldots, s$. Then, for all elements $\alpha_1 \ldots \alpha_s \in V$ (where $\alpha_i \in \mathrm{Aut}\, A_i$ for all i),

$$a_1 \ldots a_s = (a_1 \ldots a_s)^{\alpha_1 \ldots \alpha_s} = a_1^{\alpha_1} \ldots a_s^{\alpha_s},$$

hence $a_i = a_i^{\alpha_i}$ for all i. Since $V_i = V^{\rho_i}$ this makes a_i a fixed point of V_i, for $i = 1, \ldots, s$; and certainly $a_i \neq 1$ for some i.

If A is cyclic, it must have odd order if it is to have a fixed-point-free group of automorphisms.

COROLLARY 4.6. If in 4.5 A is cyclic of odd order then the action of V on A is fixed-point-free if and only if, for every $i = 1, \ldots, s$, V_i is not a p_i-group.

Proof. With the notation of 4.5, A_i is now a cyclic p_i-group and V_i is cyclic. Then the result follows from 4.5 and Lemma 3.6 of ([23]).

Now from 4.1 and 4.6 we deduce at once

COROLLARY 4.7. Suppose that A is cyclic of odd order $\prod_{i=1}^{s} p_i^{n_i}$, where $p_1, \ldots, p_s$ are distinct primes and $s, n_1, \ldots, n_s$ positive integers. Let V be a subgroup of $\mathrm{Aut}\, A$. For each $i = 1, \ldots, s$ let A_i denote the Sylow p_i-subgroup of A and V_i the group of automorphisms of A_i determined by restriction of elements of V to A_i. If, for each $i = 1, \ldots, s$, V_i is not a p_i-group then $\mathrm{Aut}(VA) = \mathrm{Hol}\, A$. (In particular, this is true whenever $VA \geq \mathrm{Dih}\, A$.)

We shall consider next how Lemma 1.1 can be applied in the present context.

LEMMA 4.8. Let W be a fixed-point-free group of automorphisms of A, $U = N_{\mathrm{Aut}\, A}(W)$ and $W \leq V \leq U$. Let $H = WA$ and $G = VA$, subgroups of $\mathrm{Hol}\, A$.

(i) If $\mathrm{Aut}\, H = UA$ and H is characteristic in G then $\mathrm{Aut}\, G = N_{\mathrm{Aut}\, H}(G) = N_U(V)A = N_{\mathrm{Aut}\, A}(V)A$.

(ii) If A is characteristic in G and either $(|W|, |A|) = 1$ or W is nilpotent then $\mathrm{Aut}\, G = N_{\mathrm{Aut}\, A}(V)A$.

Proof. (i) Since the action of W on A is fixed-point-free, $C_{\mathrm{Hol}\, A}(H) = 1$. Thus it follows from 1.1 that if H is characteristic in G then $\mathrm{Aut}\, G = N_{\mathrm{Aut}\, H}(G) = N_U(V)A$, since $\mathrm{Aut}\, H = UA$. Also $G = VA \trianglelefteq N_{\mathrm{Aut}\, A}(V)A$ and $C_{\mathrm{Hol}\, A}(G) = 1$. Therefore $\mathrm{Aut}\, G$ contains a copy of $N_{\mathrm{Aut}\, A}(V)A$, which itself contains $N_U(V)A$. Hence also $\mathrm{Aut}\, G = N_{\mathrm{Aut}\, A}(V)A$.

(ii) Now suppose that A is characteristic in G and either $(|W|, |A|) = 1$ or W is nilpotent. Then by 4.1, $\mathrm{Aut}\, H = UA$. If W is characteristic in V then clearly H is characteristic in G and the result follows from (i). But even if W is not characteristic in V, we can replace it by a subgroup which is, and reach the same conclusion. If W is nilpotent let $W^* = F(V)$, the Fitting subgroup of V. If W is not nilpotent then $(|W|, |A|) = 1$: in this case let $W^* = O_{\omega}(V)$, where ω is the set of prime divisors of $|W|$. Then in either case, W^* is characteristic in V and $W^* \geq W$, so that $C_A(W^*) = 1$; and either W^* is nilpotent or $(|W^*|, |A|) = 1$. Therefore everything holds with W^* in place of W, and the proof is complete.

In what follows we shall apply this result repeatedly with W a 'relatively prime' group of automorphisms of A. It is convenient to fix notation.

NOTATION Throughout the remainder of this paper, W will denote a fixed-point-free group of automorphisms of A such that $(|W|, |A|) = 1$, and H will denote the subgroup WA of $\mathrm{Hol}\, A$. Further, π will denote the set of prime divisors of $|A|$.

In order to apply 4.8 we need some information about the normal subgroups of H.

LEMMA 4.9 The normal subgroups of H are precisely the subgroups XB, where B is a W-invariant subgroup of A, $X \trianglelefteq W$ and $X \leq C_W(A/B)$.

Proof. Let $L \trianglelefteq H$ and let $B = L \cap A$. Then B is W-invariant and B is the normal Hall π-subgroup of L. By Schur's theorem, B has a complement X in L. Since H is π-soluble and W is a Hall π'-subgroup of H, there is an element $h \in H$ such that $X^h \leq W$ (Huppert [11] 660, VI.1.7). Now we may replace X by X^h and assume that $X \leq W$. Then $L = XB$ and $X = L \cap W \trianglelefteq W$. Furthermore $[X, A] \leq L \cap A = B$, so that $X \leq C_W(A/B)$.

Conversely, suppose that B is a W-invariant subgroup of A, $X \trianglelefteq W$ and $X \leq C_W(A/B)$. Then $B \trianglelefteq H$; and $XB/B \trianglelefteq WB/B.A/B = H/B$, hence $XB \trianglelefteq H$.

From 4.1, 4.8 and 4.9 we deduce the following criterion.

THEOREM 4.10. Let $W \le V \le N_{\mathrm{Aut}\,A}(W)$ and $G = VA \le \mathrm{Hol}\,A$. Suppose that there do not exist normal subgroups V_1 and W_1 of V and a normal subgroup X of W such that $V \ge V_1 > W_1 \ge W$ and $V_1/W_1 \cong W/X$. Then $\mathrm{Aut}\,G = N_{\mathrm{Aut}\,A}(V)A$.

Proof. Suppose that the conclusion is false. Then by 4.8 (i), H is not characteristic in G. Hence (by the argument leading to 1.2) there is a non-trivial quotient of H isomorphic to a normal subgroup of G/H, and hence to a normal subgroup of V/W. Say $L \lhd H$ and $V_1/W \trianglelefteq V/W$ with $H/L \cong V_1/W$. By 4.9, $L = XB$, where B is a W-invariant subgroup of A, $X \trianglelefteq W$ and $X \le C_W(A/B)$. Then V_1/W is isomorphic to a semi-direct product of A/B by W/X (with action induced by the action of W on A). Let W_1/W be the normal subgroup of V_1/W corresponding in this isomorphism to A/B. Then $W_1/W = O_\pi(V_1/W)$, and so $W_1 \trianglelefteq V$. Moreover, $V_1/W_1 \cong W/X$. Therefore, by hypothesis, $X = W$ and so $[W, A] \le B$. Hence by 4.1, $B = A$. But then $L = H$, a contradiction.

Now we establish another criterion.

THEOREM 4.11. Let $W \le V \le N_{\mathrm{Aut}\,A}(W)$ and $G = VA \le \mathrm{Hol}\,A$. Suppose that for each V-invariant proper subgroup B of A either $O_\pi(C_V(A/B)) = 1$ or $C_V(A/B)$ does not contain a subgroup isomorphic to W. Then $\mathrm{Aut}\,G = N_{\mathrm{Aut}\,A}(V)A$.

Proof. By 4.8 (ii) it is enough to show that A is characteristic in G. Let $\gamma \in \mathrm{Aut}\,G$. If $H^\gamma = H$ then $A^\gamma = O_\pi(H)^\gamma = O_\pi(H^\gamma) = O_\pi(H) = A$. Therefore we may assume that $H^\gamma = H^* \ne H$. The image under γ of any subgroup F of G will be denoted by F^*. Now $J = H \cap H^* \lhd G$ and so in particular $J \lhd H$. Hence by 4.9, $J = XB$ for some W-invariant $B \le A$ and some $X \trianglelefteq W$ such that $X \le C_W(A/B)$. Then $B = O_\pi(J) \lhd G$, so that B is V-invariant.

Let $\overline{G} = G/J$ and let the bar convention apply. Thus $\overline{G} = \overline{V}\,\overline{A}$. Since $\overline{H}$ and $\overline{H^*}$ are normal subgroups of $\overline{G}$ with trivial intersection,

$$\overline{H^*} \le C_{\overline{G}}(\overline{H}) \le C_{\overline{G}}(A) = C_V(\overline{A})\overline{A}.$$

A straightforward calculation shows that $C_{\overline{V}}(\overline{A}) = \overline{C_V(A/B)}$, and so

$$H^* \leq C_V(A/B)AJ = C_V(A/B)A, \qquad \text{(i)}$$

since $X \leq C_W(A/B)$. Now $A^* = O_\pi(H^*) \leq O_\pi(C_V(A/B)A)$, since $H^* \lhd G$. It is easy to show for any $T \leq V$, $O_\pi(TA) = O_\pi(T)A$. Hence $A^* \leq O_\pi(C_V(A/B))A$. (ii) By (i), $W^* \cong W^*A/A \leq C_V(A/B)A/A \cong C_V(A/B)$, so that $C_V(A/B)$ has a subgroup isomorphic to W. By hypothesis, it follows that if $B < A$ then $O_\pi(C_V(A/B)) = 1$. But then (ii) shows that $A^* = A$, as required. Therefore we may assume that $B = A$. Then $A \leq H^*$ and so $A = O_\pi(H^*) = A^*$, as required.

COROLLARY 4.12. Let $W \leq V \leq N_{\mathrm{Aut}\ A}(W)$ and $G = VA \leq \mathrm{Hol}\ A$. Then $\mathrm{Aut}\ G = N_{\mathrm{Aut}\ A}(V)A$ if any one of the following conditions holds:

(i) $O_\pi(V) = 1$, or

(ii) any normal subgroup of V which contains a copy of W contains W, or

(iii) no chief factor of V/W is isomorphic to a quotient of W.

Proof. For any V-invariant subgroup B of A, $C_V(A/B) \trianglelefteq V$, so that $O_\pi(C_V(A/B) \leq O_\pi(V)$. Moreover, when $B < A$, $C_V(A/B) \ngeq W$, by 4.1. Now cases (i) and (ii) are covered by 4.11 and case (iii) is covered by 4.10.

A special case of interest occurs when $|A|$ is odd and V is any subgroup of Aut A which contains the (fixed-point-free) automorphism which inverts every element of A; equivalently $G = VA \geq \mathrm{Dih}\ A$. Then we may for instance choose $W = \langle \eta \rangle$ (or similarly $O_2(V)$ or $O_{\pi'}(V)$).

COROLLARY 4.13. Suppose that $|A|$ is odd and that G is a subgroup of Hol A which contains Dih A. Let $V = G \cap \mathrm{Aut}\ A$, so that $G = VA$, and let η be the automorphism $a \mapsto a^{-1}$ of A. Then $\mathrm{Aut}\ G = N_{\mathrm{Aut}\ A}(V)A$ if any one of the following conditions holds:

(i) $O_\pi(V) = 1$, or

(ii) every normal subgroup of V of even order contains $\langle \eta \rangle$, or

(iii) no chief factor of $V/\langle \eta \rangle$ is of order 2.

In Example 4.15 we shall see that these conditions cannot be omitted.

Although we shall not need it in what follows, we note one more criterion, because it is the crux of G.A. Miller's proof that when $|A|$ is odd, Hol A is complete.

LEMMA 4.14. (Cf. Miller [14]). Suppose that W is abelian and let $W \leq V \leq C_{\mathrm{Aut}\, A}(W)$ and $G = VA \leq \mathrm{Hol}\, A$. Then $\mathrm{Aut}\, G = N_{\mathrm{Aut}\, A}(V)A$ unless there is a normal subgroup A^* of G such that $A \neq A^* \cong A$, $A^* = (\mathrm{Aut}\, A) \cap A^* \times A \cap A^*$, and $A \cap A^*$ has the same exponent as A.

Proof. By 4.8 (ii), if $\mathrm{Aut}\, G \neq N_{\mathrm{Aut}\, A}(V)A$, there is automorphism of G which moves A, say to A^*: then $A^* \lhd G$ and $A \neq A^*$. Since $V \leq C_{\mathrm{Aut}\, A}(W)$, $[W, G] = [W, A] \leq A$. Hence $W \leq C_V(A^*/A \cap A^*)$. Now $A \cap A^*$ is W-invariant and W fixes each element of $A^*/A \cap A^*$. Since $(|W|, |A \cap A^*|) = 1$ it follows from a result of G. Glauberman (Huppert [11] 131, I.18.6) that every coset of $A \cap A^*$ in A^* contains an element which is fixed by W. Hence $A^* = (A \cap A^*)C_{A^*}(W)$. It is easy to see that $C_{\mathrm{Hol}\, A}(W) = C_{\mathrm{Aut}\, A}(W)C_A(W) = C_{\mathrm{Aut}\, A}(W)$, by hypothesis. Hence $C_{A^*}(W) \leq (\mathrm{Aut}\, A) \cap A^*$. Therefore, since A^* is abelian and $(\mathrm{Aut}\, A) \cap A = 1$, $A^* = (\mathrm{Aut}\, A) \cap A^* \times A \cap A^*$.

Since A and A^* normalize each other $[(\mathrm{Aut}\, A) \cap A^*, A] \leq A \cap A^*$, so that $(\mathrm{Aut}\, A) \cap A^* \leq C_{\mathrm{Aut}\, A}(A/A \cap A^*)$. But also $(\mathrm{Aut}\, A) \cap A^* \leq C_{\mathrm{Aut}\, A}(A \cap A^*)$. Thus $(\mathrm{Aut}\, A) \cap A^*$ is contained in the stability group of A with respect to $A \cap A^*$. Hence (Huppert [11] 20, I.4.4) $\exp((\mathrm{Aut}\, A) \cap A^*)$ divides $\exp(A \cap A^*)$. Therefore $\exp A = \exp A^* = \exp(A \cap A^*)$.

In §5 we shall apply the criteria obtained above to the particular case when A is elementary. We end this section by setting down an example to show that $\mathrm{Aut}(VA)$ cannot always be identified with $N_{\mathrm{Aut}\, A}(V)A$ when V is a fixed-point-free group of automorphisms of A.

EXAMPLE 4.15. Let $V \leq \mathrm{Aut}\, A$ with $C_A(V) = 1$ and let $G = VA \leq \mathrm{Hol}\, A$. If $\mathrm{Aut}\, G = N_{\mathrm{Aut}\, A}(V)A$ then A is characteristic in G. We show that A and V can be chosen so that A is not characteristic in G, and hence so that

$\mathrm{Aut}\, G \neq N_{\mathrm{Aut}\, A}(V)A$.

Let A be elementary of order p^2, where $p \geq 5$, say $A = \langle a_1 \rangle \times \langle a_2 \rangle$, where $a_1^p = a_2^p = 1$. We identify Aut A with $GL(2, p)$ by identifying each automorphism of A with the matrix representing it with respect to the base a_1, a_2 of A. Let r be an integer such that $r \not\equiv 0(p)$ and $r^2 \not\equiv 1(p)$, and let $m = o(r, p)$. We use the same symbol r for the residue class (mod p) which r determines. Now let $V = \langle \theta, \zeta \rangle \leq \mathrm{Aut}\, A$, where $\theta = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $\zeta = \begin{pmatrix} r^2 & 0 \\ 0 & r \end{pmatrix}$. Then θ has order p, ζ has order m and $\zeta^{-1}\theta\zeta = \theta^r$. Hence $|V| = pm$. Since $C_A(\zeta) = 1$, $C_A(V) = 1$. Let $G = VA \leq \mathrm{Hol}\, A$. Then $G = \langle \theta, \zeta, a_1, a_2 \rangle$ with defining relations

$$\theta^p = \zeta^m = a_1^p = a_2^p = 1,\ \theta^\zeta = \theta^r,\ a_1a_2 = a_2a_1,$$
$$a_1^\theta = a_1,\ a_2^\theta = a_1a_2,\ a_1^\zeta = a_1^{r^2},\ a_2^\zeta = a_2^r.$$

Let $A^* = \langle \theta \rangle \times \langle a_1 \rangle$ and note that $A \cong A^* \lhd G$. Now there is an automorphism γ of G of order 2 such that $A^\gamma = A^*$; namely, γ is defined by the following equations, which are compatible with the defining relations of G:

$$a_1^\gamma = a_1^{-1},\ a_2^\gamma = \theta,\ \theta^\gamma = a_2,\ \zeta^\gamma = \zeta.$$

Hence A is not characteristic in G.

Now choose p and r so that m is even and let

$$\eta = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \notin V.$$

If in the construction above, we replace V by $V^* = V \times \langle \eta \rangle$ and G by $G^* = V^*A$, then $|G^*:G| = 2$ and $G^* = \langle \theta, \zeta, \eta, a_1, a_2 \rangle$ with defining relations as for G, together with the extra relations

$$\eta^2 = 1,\ \theta\eta = \eta\theta,\ \zeta\eta = \eta\zeta,\ a_1^\eta = a_1^{-1},\ a_2^\eta = a_2^{-1}.$$

Now the automorphism γ of G defined above can be extended to G^* by setting

$$\eta^\gamma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \zeta^{m/2}\eta.$$

Thus G^* is a subgroup of Hol A which contains Dih A, but which violates the conclusion of 4.13.

5. Relative holomorphs of elementary abelian groups

For the rest of the paper we consider the case in which A is elementary, of order p^n. It is convenient to view A as a vector space over GF(p), choose a particular base of A and identify each automorphism of A with the non-singular matrix over GF(p) which represents it with respect to this base. In this way we identify Aut A with GL(n, p).

We begin by noting sufficient conditions for fixed-point-free action on A of a p'-subgroup of Aut A.

LEMMA 5.1. Suppose that A is elementary, of order p^n, and let W be a p'-subgroup of Aut A. Then the action of W on A is fixed-point-free if either $|W|$ is divisible by some prime q for which $o(p, q) = n$ or W contains a Sylow q-subgroup of Aut A for some prime divisor q of p^n-1.

Proof. We may assume that $n > 1$. Let q be a prime divisor of p^n-1 such that either $o(p, q) = n$ or W contains a Sylow q-subgroup of Aut A. Let Q be a Sylow q-subgroup of W. We show that $C_A(Q) = 1$.

If $C_A(Q) \neq 1$, let $1 \neq a \in C_A(Q)$. Since p does not divide $|Q|$, Maschke's theorem implies that there is a Q-invariant subgroup B of A such that $A = \langle a \rangle \times B$. Hence Q is conjugate in GL(n, p) to a subgroup of the group of all matrices of the blocked form

$$\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & * \end{array}\right),$$

and this latter group is isomorphic to GL(n-1, p). Therefore $|Q|$ divides $|GL(n-1, p)| = (p^{n-1}-1)(p^{n-1}-p) \ldots (p^{n-1}-p^{n-2})$. This gives a contradiction if $o(p, q) = n$. We also get a contradiction if Q is a Sylow q-subgroup of

Aut A: for $|GL(n, p)|$ is divisible by $(p^n-1)|GL(n-1, p)|$ and therefore, since q divides p^n-1, $|Q|$ does not divide $|GL(n-1, p)|$.

If q is a prime such that $o(p, q) = n$ then $GL(n, p)$ has cyclic Sylow q-subgroups (Huppert [11] 187, II.7.3) and any element of $GL(n, p)$ of order q acts irreducibly on A. We shall discuss this action later on in this section.

THEOREM 5.2. Suppose that A is elementary of order p^n, and let q be a prime divisor of p^n-1 and Q a Sylow q-subgroup of Aut A. If $Q \leq V \leq N_{\mathrm{Aut}\ A}(Q) = U$, say, and $G = VA \leq \mathrm{Hol}\ A$ then $\mathrm{Aut}\ G = N_{\mathrm{Aut}\ A}(V)A = N_U(V)A$. In particular, the subgroup UA of Hol A is complete.

Proof. By 5.1, Q is a fixed-point-free group of automorphisms of A. Then we may take $W = Q$ in 4.12 and deduce that $\mathrm{Aut}\ G = N_{\mathrm{Aut}\ A}(V)A$. Now A is characteristic in G, and certainly Q is characteristic in V. Hence QA is characteristic in G and therefore, by 4.8 (i), $\mathrm{Aut}\ G = N_U(V)A$.

THEOREM 5.3. Suppose that A is elementary of order p^n, and let V be a subgroup of Aut A which acts irreducibly on A. Let $G = VA \leq \mathrm{Hol}\ A$. If $O_{p'}(V) \neq 1$ then $\mathrm{Aut}\ G = N_{\mathrm{Aut}\ A}(V)A$.

Proof. Let $W = O_{p'}(V) \neq 1$, and let $a \in C_A(W)$. Since $W \trianglelefteq V$, $a^\alpha \in C_A(W)$ for all $\alpha \in V$. Therefore W fixes every point of $\langle a^\alpha : \alpha \in V\rangle = A$ if $a \neq 1$, since V acts irreducibly on A. This contradicts $W \neq 1$. Hence the action of W on A is fixed-point-free and we may apply 4.11: the only V-invariant proper subgroup of A is 1, and so $\mathrm{Aut}\ G = N_{\mathrm{Aut}\ A}(V)A$.

If in 5.3, V is p-soluble then automatically $O_{p'}(V) \neq 1$: for, since V has a faithful, irreducible representation in characteristic p, $O_p(V) = 1$.

COROLLARY 5.4. Let A be elementary of order p^n and let V be any p-soluble subgroup of Aut A which acts irreducibly on A. Then $\mathrm{Aut}(VA) = N_{\mathrm{Aut}\ A}(V)A$.

If in 5.4 V is abelian then in fact V is cyclic, $|V|$ divides p^n-1 and

$n = o(p, |V|)$ (Huppert [11] 165, II.3.10). Moreover, the action of a generator of V on A is equivalent to the action by multiplication of an element of $GF(p^n)^{\times}$ of order $|V|$ on $GF(p^n)^{+} \cong A$. Conversely, the action by multiplication of any non-trivial subgroup of $GF(p^n)^{\times}$ on $GF(p^n)^{+}$ is faithful and fixed-point-free; and is actually irreducible if $n = o(p, m)$, where m is the order of the subgroup. We shall now investigate this action.

Let A be elementary of order p^n with $n > 1$ and let ζ be an element of Aut A of order p^n-1 corresponding to a generator of $GF(p^n)^{\times}$ acting on $GF(p^n)^{+}$: ζ is called a Singer cycle (Huppert [11] 187, II.7.3). Let $T = \langle\zeta\rangle$ and $S = N_{\mathrm{Aut}\,A}(T)$. Then $C_{\mathrm{Aut}\,A}(\zeta) = T$ and $S = \langle\alpha\rangle T$, where $\alpha^n = 1$ and $\zeta^{\alpha} = \zeta^p$ (Huppert, loc. cit.). Every non-trivial subgroup of T has fixed-point-free action on A. Since $(p^n-1, p^n) = 1$, it follows at once from 4.1 that $\mathrm{Aut}(\langle\zeta^j\rangle A) = N_{\mathrm{Aut}\,A}(\langle\zeta^j\rangle)A$ for every divisor j of p^n-1 with $1 \leq j < p^n-1$. In particular, $\mathrm{Aut}(TA) = SA$. Moreover, if $o(p, p^n-1/j) = n$ then $N_{\mathrm{Aut}\,A}(\langle\zeta^j\rangle) = S$ (Huppert, loc. cit.).

Now suppose that $T < V \leq S$. Since T acts irreducibly on A, so also does V. Moreover, $1 \neq T \leq O_{p'}(V)$. Hence 5.3, $\mathrm{Aut}(VA) = N_{\mathrm{Aut}\,A}(V)A$. Certainly $S \leq N_{\mathrm{Aut}\,A}(V) \leq N_{\mathrm{Aut}\,A}(V')$. We prove that $N_{\mathrm{Aut}\,A}(V) = S$ by showing that $N_{\mathrm{Aut}\,A}(V') = S$. We have $V = \langle\alpha^{\ell}, \zeta\rangle$ for some proper divisor ℓ of n. Since every subgroup of T is normal in S, $V' = \langle[\zeta, \alpha^{\ell}]\rangle = \langle\zeta^{p^{\ell}-1}\rangle$. By the result quoted above, $N_{\mathrm{Aut}\,A}(V') = S$ provided $o(p, p^n-1/p^{\ell}-1) = n$. Suppose to the contrary that $o(p, p^n-1/p^{\ell}-1) = m < n$. Then m divides n and

$$(p^{\ell}-1)(p^m-1) = k(p^n-1)$$

for some positive integer k. If $\ell \leq m$ then

$$p^{\ell}-1 = k(1 + p^m + p^{2m} + \dots + p^{(n/m-1)m})$$

$$\geq 1 + p^m, \text{ since } m < n.$$

This is a contradiction. Hence $\ell > m$. But then the same argument with ℓ and m interchanged leads again to a contradiction.

In summary we have proved

THEOREM 5.5. Suppose that A is elementary of order p^n, where $n > 1$. Let ζ be a Singer cycle in Aut A, $T = \langle\zeta\rangle$ and $S = N_{\mathrm{Aut}\,A}(T) = \langle\alpha\rangle T$, where $\alpha^n = 1$ and $\zeta^\alpha = \zeta^p$. Then

(i) For every proper divisor j of p^n-1, $\mathrm{Aut}(\langle\zeta^j\rangle A) = N_{\mathrm{Aut}\,A}(\langle\zeta^j\rangle)A$. In particular, $\mathrm{Aut}(TA) = SA$. Moreover, $\mathrm{Aut}(\langle\zeta^j\rangle A) = SA$ whenever $o(p, p^n-1/j) = n$.

(ii) For every divisor m of n, $\mathrm{Aut}(\langle\alpha^m, \zeta\rangle\, A) = SA$. In particular, SA is complete.

The groups TA and SA are easily recognizable. Since the action of T on A is equivalent to the action of $GF(p^n)^{\times}$ on $GF(p^n)^{+}$ by multiplication, one sees that TA is isomorphic to the group of affine transformations of $GF(p^n)$, that is the group of maps $GF(p^n) \to GF(p^n)$ of the form

$$x \mapsto ax + b,$$

where $a \in GF(p^n)^{\times}$ and $b \in GF(p^n)^{+}$. This group will be denoted by $\mathcal{A}(p^n)$. Also the action by conjugation of $\langle\alpha\rangle$ on T is equivalent to the action of the group $\mathrm{Aut}\,GF(p^n)$ of field automorphisms of $GF(p^n)$ on $GF(p^n)^{\times}$. Furthermore the action of $S = \langle\alpha\rangle T$ on A is the natural one (see [11], 188, proof of II.7.3a). Thus SA is isomorphic to the group of maps $GF(p^n) \to GF(p^n)$ of the form

$$x \mapsto ax^{\alpha} + b,$$

where $a \in GF(p^n)^{\times}$, $b \in GF(p^n)^{+}$, $\alpha \in \mathrm{Aut}\,GF(p^n)$. We shall follow R.S. Dark ([3]) in calling this group the extended affine group of $GF(p^n)$ (often called the group of semilinear transformations of $GF(p^n)$) and denote this by $\mathcal{J}(p^n)$.

Combining 5.5 with known facts about $\mathcal{A}(p) = \mathcal{J}(p)$, we have

COROLLARY 5.6. Providing that $p^n \neq 2$, the affine group $\mathcal{A}(p^n)$ of $GF(p^n)$ has its automorphism group isomorphic to the extended affine group $\mathcal{J}(p^n)$ of $GF(p^n)$, and $\mathcal{J}(p^n)$ is complete.

Note. The fact that Aut $\mathcal{A}(p^n) \cong \mathcal{J}(p^n)$ is contained in Lemma 8 of R.S. Dark ([3]).

We retain the notation of 5.5 and consider the possibility that SA has complete proper subgroups among the relative holomorphs of A. Any such subgroup if of the form VA, where V is a self-normalizing subgroup of S.

We can characterize the abnormal subgroups of S. If V is such a subgroup then $VS' = S$ and $V \cap S' \trianglelefteq S$. Then $V/V \cap S'$ is a complement to $S'/V \cap S'$ in $S/V \cap S'$. All such complements are conjugate, because they are Carter subgroups of $S/V \cap S'$.

Thus we ask: if $Y \leq S'$ (and therefore, since S' is cyclic, $Y \trianglelefteq S$), when does S/Y split over S'/Y? If it does then so also must T/Y split over S'/Y. Since T/Y is cyclic, this splitting occurs if and only if $(|T/S'|, |S'/Y|) = 1$; or equivalently, since $S' = \langle \zeta^{p-1} \rangle$, if and only if $(p-1, |S'/Y|) = 1$.

Suppose now that $Y \leq S'$ with $|S'/Y| = j$ and $(p-1, j) = 1$. Then $Y = \langle \zeta^{(p-1)j} \rangle$. Let $\overline{S} = S/Y = \langle \overline{\alpha} \rangle \langle \overline{\zeta} \rangle$. Here $\overline{\zeta}$ has order $(p-1)j$, and since $(p-1, j) = 1$, $\langle \overline{\zeta} \rangle = \langle \overline{\zeta}^j \rangle \times \langle \overline{\zeta}^{p-1} \rangle$. Now $\langle \overline{\alpha} \rangle \langle \overline{\zeta}^j \rangle$ is a subgroup of $\overline{S}$ which complements $\langle \overline{\zeta}^{p-1} \rangle = \overline{S}'$ in $\overline{S}$. Let $V = \langle \alpha \rangle \langle \zeta^j \rangle$. Then $VS' = S$ and $V \cap S' = Y$. The normal closure of V in S is S and so ([21] Corollary 1.5) V is abnormal in S. This establishes

LEMMA 5.7. With the hypotheses and notation of 5.5, the number of conjugacy classes of abnormal subgroups of S (including S itself) is the number of divisors j of $|S'| = p^n-1/p-1$ such that $(j, p-1) = 1$. For each such j, a representative of the corresponding class of abnormal subgroups is $\langle \alpha, \zeta^j \rangle$, of order $n(p^n-1)/j$.

We now consider a typical abnormal subgroup $V = \langle \alpha, \zeta^j \rangle$ of S, where j divides $p^n-1/p-1$ and $(j, p-1) = 1$. If $p = 2$ and $V = \langle \alpha \rangle$ then the action of V on A is not fixed-point-free: for if we identify A with $GF(2^n)^+$ we see that α fixes 1 as well as 0. We discard this case. In every other case, $\zeta^j \neq 1$ and so $C_A(\zeta^j) = 1$. Then $Z(VA) = 1$.

Let $W = O_{p'}(V) \geq \langle \zeta^j \rangle \neq 1$. Since $V/\langle \zeta^j \rangle$ is cyclic, we see that V/W is a p-group. Hence by 4.12, $\mathrm{Aut}(VA) = N_{\mathrm{Aut}\,A}(V)A$. Now $V' = \langle [\zeta^j, \alpha] \rangle = \langle \zeta^{(p-1)j} \rangle$, and $N_{\mathrm{Aut}\,A}(V) \leq N_{\mathrm{Aut}\,A}(V')$. By a result quoted previously, $N_{\mathrm{Aut}\,A}(V') = S$ if $o(p, p^n-1/(p-1)j) = n$, and then $N_{\mathrm{Aut}\,A}(V) = N_S(V) = V$. Then VA is complete.

COROLLARY 5.8. With the hypotheses and notation of 5.5, for each divisor j of $p^n-1/p-1$ such that $(j, p-1) = 1$, $V = \langle \alpha, \zeta^j \rangle$ is abnormal in S and (excluding the case $p = 2$, $j = 2^n-1$) $\mathrm{Aut}(VA) = N_{\mathrm{Aut}\,A}(V)A$. Moreover, VA is complete if $o(p, p^n-1/(p-1)j) = n$.

Consider the simplest non-trivial case: $n = 2$ with $p > 2$. Then $p^n-1/p-1 = p+1$; and since $(p+1, p-1) = 2$, the divisors j of $p+1$ such that $(j, p-1) = 1$ are just the odd divisors of $p+1$. Consider such a divisor j, and suppose that $o(p, p+1/j) \neq 2$. Then $o(p, p+1/j) = 1$, that is there is a positive integer k such that $j(p-1) = k(p+1)$. Then $j(p-1/2) = k(p+1/2)$, and since $(p-1/2, p+1/2) = 1$, j is divisible by $p+1/2$. But $j \leq p+1/2$. Hence $o(p, p+1/j) = 2$ unless $j = p+1/2$. Thus, when $n = 2$ and $p > 2$, all the groups VA mentioned in 5.8, one for each odd divisor j of $p+1$, are complete; except possibly for $j = p+1/2$.

From this we now prove

COROLLARY 5.9. Let m be any positive integer and $(q_1, \ldots, q_m)$ any sequence of m odd primes (not necessarily distinct). Let k be any integer > 2 and let p be any prime which is congruent to $-1 \pmod{kq_1q_2 \ldots q_m}$. (Such a p exists, by Dirichlet's theorem.) Then in the extended affine group $\mathcal{J}(p^2) = G$, say, there is a chain of subgroups $G = G_0 > G_1 > \ldots > G_m$

with $|G_{i-1}:G_i| = q_i$ for $i = 1, \ldots, m$, and every G_i complete $(i = 0, 1, \ldots, m)$.

Proof. As before we identify $\mathcal{f}(p^2)$ with the group SA, where the notation is as in 5.5 and $n = 2$. For each $i = 1, \ldots, m$ let $V_i = \langle \alpha, \zeta^{j_i} \rangle$, where $j_i = q_1 q_2 \ldots q_i$, and let $G_i = V_i A$. Then $|G_o| = |G| = 2(p^2-1)p^2$; and for each $i > 0$, $|G_i| = 2(p^2-1)p^2/j_i$. Hence $|G_{i-1}:G_i| = q_i$ for all i. Since j_i is an odd divisor of $p + 1$, V_i is abnormal in S. Moreover, G_o is complete; and for each $i > 0$, $j_i \leq j_m$, which divides $p+1 / k < p+1 / 2$, hence G_i is complete.

We note finally that a group VA appearing in 5.8 need not be complete. If for instance $p = 5$ and $n = 2$ then $p^n-1 / p-1 = 6$. The only odd divisors of 6 are 1 and 3. Here $3 = p+1 / 2$, the possibly exceptional case in the argument above. Consider $V = \langle \alpha, \zeta^3 \rangle$, an abnormal subgroup of S. Certainly $Z(VA) = 1$. Now $|V| = 2(5^2-1)/3 = 2^4$ and $|GL(2, 5)| = 2^5.3.5$. In this case V is contained as a proper normal subgroup in a Sylow 2-subgroup of Aut A. Thus $VA < N_{\mathrm{Aut}\ A}(V)A = \mathrm{Aut}(VA)$.

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Mathematics Department,

University of Newcastle upon Tyne.

C 17

Frattini normal subgroups of finite groups

Let G be a finite group, A a normal subgroup of G and H a subgroup of G containing A and such that $(|G:H|, |A|) = 1$. Then, according to a well known theorem of W. Gaschütz [1],

(I). Providing that A is abelian, G splits over A if and only if H splits over A.

There are in the literature examples to show that the splitting of G over A does not follow from the splitting of H over A if in (I) the hypothesis that A is abelian is weakened to 'A is nilpotent': see examples of H.J. Zassenhaus ([8], pp.243-4), R. Baer ([7], pp.225-6) and K.H. Hoffmann ([4], I. 18.7, p.131).

In his paper, Gaschütz proved in parallel with (I) a 'partial splitting' result, which may be formulated as

(II). Providing that A is abelian, $A \le \Phi(G)$ if and only if, whenever S is a supplement to A in H, $(S \cap A)^G = A$.

Here $\Phi(G)$ denotes the Frattini subgroup of G, and for any subgroup L of G, L^G denotes the normal closure of L in G, that is the smallest normal subgroup of G which contains L. A subgroup S of H is a supplement to A in H if $SA = H$.

We shall show that in contrast to (I), the hypothesis on A in (II) can be weakened without invalidating the conclusion. This is proved by means of a simple induction argument, using (I).

Theorem 1. Let A be a nilpotent normal subgroup of the finite group G and let $A \leq H \leq G$ with $(|G : H|, |A|) = 1$. Then $A \leq \Phi(G)$ if and only if, for every supplement S to A in H, $(S \cap A)^G = A$.

Proof. Suppose first that $A \nleq \Phi(G)$. Then there is a maximal subgroup M of G such that $G = MA$. Hence $H = (M \cap H)A$ and $M \cap H \cap A = M \cap A$, which is normal in M. Moreover, since A is nilpotent and $M \cap A < A$, $M \cap A \ntrianglelefteq N_A(M \cap A)$. Therefore, since M is maximal in G, $M \cap A$ is normal in G. Thus $M \cap H$ is a supplement to A in H and $(M \cap H \cap A)^G = M \cap A < A$.

Now suppose that $A \leq \Phi(G)$ and argue by induction on $|A|$. If $|A| = 1$ there is nothing to prove, so assume that $|A| > 1$ and let S be a supplement to A in H. Let B be any non-trivial normal subgroup of G contained in A, and, for every subgroup L of G, let $\bar{L} = LB/B$. Then

$$\bar{A} \leq \overline{\Phi(G)} = \Phi(\bar{G}) \quad \text{(since } B \leq \Phi(G))$$

and

$$\bar{A} \leq \bar{H} \leq \bar{G} \quad \text{with } (|\bar{G} : \bar{H}|, |\bar{A}|) = 1.$$

Since $\bar{A}$ is a nilpotent normal subgroup of $\bar{G}$ and $\bar{S}\bar{A} = \bar{H}$, the induction hypothesis implies that

$$(\bar{S} \cap \bar{A})^{\bar{G}} = \bar{A}.$$

Hence $((SB) \cap A)^G = A$,

that is $(S \cap A)^G B = A$. (*)

Now if $S \cap A = 1$ then (*) implies that A is a minimal normal subgroup of G and therefore, since A is nilpotent, that A is abelian. Moreover, H splits over A, and so it follows from Gaschütz's splitting theorem (I) that G splits over A. But this is in contradiction to $1 \neq A \leq \Phi(G)$. Therefore $S \cap A \neq 1$ and so in (*) we may choose $B \leq (S \cap A)^G$ and conclude that $(S \cap A)^G = A$.

Of course, Theorem 1 would look neater and more obviously analogous to (I) if it were possible to formulate the conclusion as '$A \leq \Phi(G)$ if and only if $A \leq \Phi(H)$'. Now it is certainly true that if $A \leq \Phi(H)$ then $A \leq \Phi(G)$: this follows from Theorem 1 and, indeed, is true whenever A is a normal subgroup of a finite group G and $A \leq H \leq G$, by another result of Gaschütz ([2], Satz 5, or [4], III.3.3a), p.269). However, as we shall see in Theorem 4, it is not in general true, under the hypotheses of Theorem 1, that if $A \leq \Phi(G)$ then $A \leq \Phi(H)$. We note some special circumstances in which this implication does hold.

Theorem 2. Let A be a normal subgroup of the finite group G, let $A \leq H \leq G$ with $(|G : H|, |A|) = 1$, and suppose that $A \leq \Phi(G)$. If $\Phi(H)$ is normal in G (in particular, if H is normal in G) then $A \leq \Phi(H)$.

Proof. Suppose the result false and let G provide a counter-example of least possible order. Then $A \neq 1$. Suppose that $\Phi(H) \neq 1$ and, for each subgroup L of G, let $\bar{L} = L\Phi(H)/\Phi(H)$. Then $\bar{A}$ is normal in $\bar{G}$, $\bar{A} \leq \bar{H} \leq \bar{G}$ and $(|\bar{G} : \bar{H}|, |\bar{A}|) = 1$. Moreover, $\bar{A} \leq \overline{\Phi(G)} \leq \Phi(\bar{G})$ and $\Phi(\bar{H}) = 1$. Hence, by the minimality of G, $\bar{A} = 1$: that is $A \leq \Phi(H)$, contrary to hypothesis. Therefore $\Phi(H) = 1$. Since A is normal in H it follows that $\Phi(A) = 1$ ([4], III.3.3b), p.269); hence, since $A \leq \Phi(G)$, that A is abelian (Gaschütz [2], Satz 9, or [4], III.3.11, p.271). Then, since $A \cap \Phi(H) = 1$, H splits over A ([2], Satz 7, or [4], III.4.4, p.278). But then, by Gaschütz's splitting theorem (I), G splits over A, in contradiction to $1 \neq A \leq \Phi(G)$.

In the rest of this paper, p always denotes a prime number. For any finite group G, $O_p(G)$ and $O_{p'}(G)$ denote, respectively, the largest normal p-subgroup of G and the largest normal subgroup of G of order not divisible by p, and $O_{p,p'}(G)$ is defined by the equation $O_{p'}(G/O_p(G)) = O_{p,p'}(G)/O_p(G)$.

A case of particular interest in (I) and (II) occurs when A is a normal p-subgroup of G for some p and H is a Sylow p-subgroup of G. The remaining results in this paper are concerned with this situation.

Theorem 3. Let A be a normal p-subgroup of the finite group G and let P be a Sylow p-subgroup of G. If $A \leq \Phi(G) \cap Z(P)$ then $A \leq \Phi(P)$.

Proof. Suppose the result false and let G provide a counter-example of least possible order. Then $A \neq 1$. Let B be a minimal normal subgroup of G with $B \leq A$. Then $B \not\leq \Phi(P)$: for if $B \leq \Phi(P)$ then $\Phi(P/B) = \Phi(P)/B \not\geq A/B$ and $A/B \leq \Phi(G)/B \cap Z(P)/B \leq \Phi(G/B) \cap Z(P/B)$, so that G/B would provide a counter-example to the theorem of smaller order than G.

Suppose that $B \leq \Phi(H)$ for some $H < G$ and let P_1 be a Sylow p-subgroup of H. Then $P_1 \leq P^x$ for some $x \in G$. Since B is normal in G and $B \leq Z(P)$, it follows that $B \leq Z(P^x)$, hence also $B \leq Z(P_1)$. Then, by the minimality of G, the assertion of the theorem is correct for H, so that $B \leq \Phi(P_1) \leq \Phi(P^x)$ (since P^x is a p-group and by [4], III.3.3b), p.269). But from this it follows that $B \leq \Phi(P)$, which we have shown to be false. Thus, for every proper subgroup H of G, $B \not\leq \Phi(H)$.

Certainly $P < G$. Let M be any maximal subgroup of G containing P and let S be a minimal supplement to B in M. Then $S \cap B$ is a normal p-subgroup of S and $S \cap B \leq \Phi(S)$ (see [4] III.3.9, p.271). Moreover, $S \cap B \leq S \cap Z(P) \leq Z(S \cap P)$. Now $B \leq P \leq M = SB$, so that $SP = M$. Hence $|S : S \cap P| = |M : P|$ and therefore $S \cap P$ is a Sylow p-subgroup of S.

Thus since $S < G$ and by the minimality of G, $S \cap B \leq \Phi(S \cap P)$. Hence (since P is a p-group) $S \cap B \leq \Phi(P)$.

Furthermore, since B is abelian, $S \cap B$ is normal in $SB = M$; and since $B \not\leq \Phi(M)$, $S < M$, so that $S \cap B < B$. If $S \cap B$ were normal in G then, since B is minimal normal in G, we should have $S \cap B = 1$. But then M would split over B, hence also P would split over B, and so, by Gaschütz's splitting theorem (I), G would split over B, in contradiction to $1 \neq B \leq \Phi(G)$. Hence $N_G(S \cap B) = M$.

For any $g \in G$, S^g is a minimal supplement to B in the maximal subgroup M^g of G. Since $S^g \cap B = (S \cap B)^g$, we have $N_G(S^g \cap B) = M^g$. But since $B \leq Z(P)$, $S^g \cap B$ is normal in P and therefore $P \leq M^g$. Therefore the previous arguments are applicable to M^g and S^g in place of M and S. We conclude that $S^g \cap B \leq \Phi(P)$ for every $g \in G$.

Since $1 \neq S \cap B$ and B is minimal normal in G, $(S \cap B)^G = B$. But this means that $B = \langle S^g \cap B : g \in G \rangle \leq \Phi(P)$, a contradiction.

We now show that the condition $A \leq Z(P)$ cannot be omitted in Theorem 3 by proving

Theorem 4. Let H be a finite group with cyclic Sylow p-subgroups for some prime divisor p of $|H|$ and such that H has no normal subgroup of prime index q dividing $p(p-1)$. Then there is a finite group G with an abelian normal p-subgroup $A \leq \Phi(G)$ such that $G/A \cong H$ and $A \not\leq \Phi(P)$, where P is any Sylow p-subgroup of G.

For the proof we shall use another fundamental result of Gaschütz [3]:

(III). For any finite group H and prime divisor p of $|H|$ there is a finite group G with an elementary abelian normal p-subgroup A such that $1 \neq A \leq \Phi(G)$ and $G/A \cong H$.

Proof of Theorem 4. In (III) choose H to be the given group, p the given prime, and let G be a group with the properties specified in (III). Let P be a Sylow p-subgroup of G and suppose that $A \leq \Phi(P)$. Then, since P/A is a Sylow p-subgroup of G/A and is therefore by hypothesis cyclic, P is cyclic. Hence, since A is elementary, $|A| = p$. Therefore $G/C_G(A)$ is cyclic of order dividing p-1. Since $A \leq C_G(A)$ it follows by hypothesis that

$$A \leq Z(G). \tag{i}$$

If $A \not\leq G'$ then p divides $|G/G'|$ and so G has a normal subgroup M with $|G/M| = p$. But then $A \leq \Phi(G) \leq M$, so that this contradicts the hypothesis on H. Hence

$$A \leq G'. \tag{ii}$$

Then, from (i) and (ii),

$$A \leq P \cap G' \cap Z(G) = 1,$$

since P is abelian ([4], IV.2.2, p.416). This contradicts the fact that from (III), $A \neq 1$. Thus we conclude that $A \not\leq \Phi(P)$.

In Theorem 4, we may for instance choose H to be the alternating group of degree 5 and p = 3 or 5. Now we note that any group H which satisfies the hypotheses of Theorem 4 is necessarily insoluble. This follows from

Proposition 5. Let H be a finite p-soluble group with cyclic Sylow p-subgroups for some prime divisor p of $|H|$. Then H has a normal subgroup J with $|H/J| = q$, where q is some prime dividing p(p-1).

Proof. We argue by induction on $|H|$. We may assume that $|H| > p$. Let L be a minimal normal subgroup of H. If p divides $|H/L|$ then, by the induction hypothesis, H/L has a normal subgroup of prime index dividing p(p-1), and therefore so has H. Now suppose that p does not divide $|H/L|$. Then L contains a Sylow p-subgroup of H. Since H is p-soluble, it follows that L is itself a p-group, hence by hypothesis is cyclic. Then, since L is minimal normal in H, $|L| = p$. Hence $H/C_H(L)$ is cyclic of order dividing p-1. If $C_H(L) < H$ it follows that H has a normal subgroup of prime index dividing p-1. Therefore we may suppose that $L \leq Z(H)$. Then, since p does not divide $|H/L|$, Schur's Theorem shows that L is a direct factor of H, hence that H has a normal subgroup of index p.

There remains the question: if in Theorem 3 G is p-soluble can the conclusion $A \leq \Phi(P)$ be drawn without the condition $A \leq Z(P)$? I do not know the answer. We shall prove two particular positive results in this direction.

Note that if A is a normal p-subgroup of the finite group G with $A \leq \Phi(G)$ and P is a Sylow p-subgroup of G then $A \leq P \cap \Phi(G)$. Moreover, since $P \cap \Phi(G)$ is the unique Sylow p-subgroup of $\Phi(G)$, $P \cap \Phi(G)$ is normal in G. Note also that if $P \cap \Phi(G) \leq \Phi(P)$ then in fact $P \cap \Phi(G)$ is the largest normal subgroup of G contained in $\Phi(P)$ (cf. [4], Satz III.3.3a),p.269).

Theorem 6. If G is a finite p-soluble group of p-length 1 and P is a Sylow p-subgroup of G then $P \cap \Phi(G) \leq \Phi(P)$.

Proof. Let $A = P \cap \Phi(G)$. We argue by induction on $|G|$. We may assume that $|G| > p$. Let $M = O_{p'}(G)$. Suppose first that $M \neq 1$. For each subgroup L of G let $\bar{L} = LM/M$. Then $\bar{A}$ is a normal p-subgroup of $\bar{G}$ and $\bar{A} \leq \overline{\Phi(G)} \leq \Phi(\bar{G})$. Hence, by the induction hypothesis, $\bar{A} \leq \Phi(\bar{P})$. Since p does not divide $|M|$, $\bar{P} \cong P$ and so $\Phi(\bar{P}) \cong \Phi(P) \cong \overline{\Phi(P)} \leq \Phi(\bar{P})$. Hence $\Phi(\bar{P}) = \overline{\Phi(P)}$, and so we have $A \leq \Phi(P)M$. Then A is a normal p-subgroup of $\Phi(P)M$ and $\Phi(P)$ is a Sylow p-subgroup of $\Phi(P)M$. Therefore $A \leq \Phi(P)$. Now suppose that $M = 1$. Since G has p-length 1, it follows that P is normal in G. Then, by Theorem 2, $A \leq \Phi(P)$.

Essentially the same result was proved by B. Huppert: see [5], Satz 5, and the remarks following Definition 2.

Corollary 7. Let G be a finite p-soluble group and $\mathbf{P}$ a Sylow p-subgroup of G. If $\mathbf{P}' \leq \mathbf{P} \cap \Phi(G)$ then $\mathbf{P} \cap \Phi(G) \leq \Phi(\mathbf{P})$.

Proof. Let $A = \mathbf{P} \cap \Phi(G)$. Then G/A is p-soluble with abelian Sylow p-subgroups. We may suppose $\mathbf{P} \neq 1$. Then G/A has p-length 1 ([4], VI.6.6a), p.691). Therefore, since $A \leq \Phi(G)$, G has p-length 1 ([4], VI.6.4e), p.689). Now the result follows from Theorem 6.

Theorem 8. Let G be a finite p-soluble group and P a Sylow p-subgroup of G. If every subgroup of P can be generated by at most 3 elements then $\mathbf{P} \cap \Phi(G) \leq \Phi(\mathbf{P})$.

Proof. Suppose the result false and let G provide a counter-example of least possible order. Let $A = \mathbf{P} \cap \Phi(G) \neq 1$.

(i) $O_{p'}(G) = 1$ and no non-trivial normal subgroup of G lies in $\Phi(\mathbf{P})$.

Let K be any non-trivial normal subgroup of G and, for every subgroup L of G, let $\bar{L} = LK/K$. Then $\bar{A}$ is a normal p-subgroup of $\bar{G}$, $\bar{A} \leq \overline{\Phi(G)} \leq \Phi(\bar{G})$ and $\bar{\mathbf{P}}$ is a Sylow p-subgroup of $\bar{G}$. Clearly $\bar{\mathbf{P}}$ satisfies the same hypothesis as $\mathbf{P}$. Hence, by the minimality of G, $\bar{A} \leq \Phi(\bar{P})$. If $K = O_{p'}(G)$ it follows, just as in the proof of Theorem 6, that $A \leq \Phi(\mathbf{P})$. This is contrary to hypothesis, so we conclude that $O_{p'}(G) = 1$. Again, if $K \leq \Phi(\mathbf{P})$ then $\Phi(\bar{P}) = \Phi(P)/K$ and so $A \leq \Phi(\mathbf{P})$, contrary to hypothesis.

(ii) It follows immediately that $O_p(G) = F(G)$, the Fitting subgroup of G, and $A = \Phi(G)$.

(iii) For every proper normal subgroup H of G, $\Phi(H) = 1$, and $O_p(G)$ is elementary abelian.

Certainly $\mathbf{P} \cap H$ is a Sylow p-subgroup of H and $\Phi(H) \leq \Phi(G)$ ([4], III.3.3b), p.269). Then by (ii), $\Phi(H)$ is a p-group. Since

$H < G$ and by the minimality of G, it follows that $\Phi(H) \leq \Phi(P \cap H) \leq \Phi(P)$ (since P is a p-group and by [4], loc.cit.). Then, since $\Phi(H)$ is normal in G, it follows from (i) that $\Phi(H) = 1$. Certainly $O_p(G)$ is a proper normal subgroup of G. Hence $\Phi(O_p(G)) = 1$ and therefore $O_p(G)$ is elementary abelian ([2], Satz 9).

(iv) $A < O_p(G)$.

Since $O_p(G) < G$ and G is p-soluble, $O_p(G) < O_{p,p'}(G)$. By Schur's Theorem, $O_{p,p'}(G)$ splits over $O_p(G)$ and the complements to $O_p(G)$ in $O_{p,p'}(G)$ form a single conjugacy class of subgroups of $O_{p,p'}(G)$. Let Q be such a complement. Then the Frattini argument shows that $G = N_G(Q)O_{p,p'}(G) = N_G(Q)O_p(G)$. Hence if $A = O_p(G)$, Q would be a non-trivial normal subgroup of G with $Q \leq O_{p'}(G)$, in contradiction to (i).

(v) Let B be a minimal normal subgroup of G with $B \leq A$. Then $B \not\leq \Phi(H)$ whenever $B \leq H < G$.

By (i), $B \not\leq \Phi(P)$. Then the assertion follows as in the proof of Theorem 3.

(vi) Let M be a maximal subgroup of G containing P and let S be a minimal supplement to B in M. Then $S \cap B \leq \Phi(P)$ and $N_G(S \cap B) = M$.

The argument is as in the proof of Theorem 3.

(vii) $P^G = G$.

If $P^G < G$ we could in (vi) choose $M \geq P^G$. Then $M^g \geq P$ for every $g \in G$. Since S^g is a minimal supplement to B in the maximal subgroup M^g of G, it would follow from (vi) that $S^g \cap B \leq \Phi(P)$ for every $g \in G$. Then $(S \cap B)^G = \langle S^g \cap B : g \in G \rangle \leq \Phi(P)$. By (i), this would imply that $S \cap B = 1$, in contradiction to (vi).

(viii) $|O_p(G)| = p^3$ and $|A| = p^2$.

By (iii) and the hypothesis on P, $|O_p(G)| \leq p^3$. If $|A| = p$ then $G/C_G(A)$ is cyclic of order dividing $p-1$, hence by (vii), $A \leq Z(G)$. But then $A \leq Z(P)$ and so, by Theorem 3, $A \leq \Phi(P)$, contrary to hypothesis. Therefore $|A| > p$ and so by (iv) we must have $|A| = p^2$ and $|O_p(G)| = p^3$.

(ix) $O_p(G)/A \leq Z(G/A)$.

By (viii), $|O_p(G)/A| = p$. Then the assertion follows from (vii) by the same argument as in (viii).

(x) The final contradiction.

By Schur's Theorem and (ix), $O_p(G)/A$ is a direct factor of $O_{p,p'}(G)/A$: say

$$O_{p,p'}(G)/A = O_p(G)/A \times R/A.$$

Then $R/A = O_{p'}(G/A)$, so that R is normal in G; and $A < R$. The same argument as in (iv) (with R in place of $O_{p,p'}(G)$ and A in place of $O_p(G)$) shows that $G = N_G(Q)A$, where Q is a complement to A in R. Then since $A = \Phi(G)$, Q is normal in G and so $1 \neq Q \leq O_{p'}(G)$, in contradiction to (i).

Finally, we point out an application of Corollary 7 to Gaschütz's results in [3]. First we give a brief summary of the relevant general theory (which is the justification of the statement (III) above).

Let H be a finite group and p a prime divisor of $|H|$. Suppose that H can be generated by a set of at most n elements, where n is a positive integer, and consider the class $\mathcal{C}$ of all group extensions

$$E : 1 \to A \to G \to H \to 1,$$

where A is an elementary abelian finite p-group and G is an n-generator group. Then there is in $\mathcal{C}$ an extension

$$1 \to A^* \to G^* \to H \to 1$$

which is universal for $\mathcal{C}$ in the sense that for any $E \in \mathcal{C}$ there is an epimorphism $\chi : G^* \to G$ such that the following diagram commutes:

$$\begin{tikzcd}
& A^* \arrow[r] \arrow[d] & G^* \arrow[dr] \arrow[d, "\chi"] & & \\
1 \arrow[ur] \arrow[dr] & & & H \arrow[r] & 1 ; \\
& A \arrow[r] & G \arrow[ur] & &
\end{tikzcd}$$

and $|A^*| = p^{1+(n-1)|H|}$. (See [3], Satz 1, and [6], Theorem 3.)

Now Gaschütz shows ([3], Satz 6) that A^* is expressible as a direct product of normal subgroups of G^*, say $A^* = B^* \times C^*$, such that $A^*/B^* \leq \Phi(G^*/B^*)$ and G^*/C^* splits over A^*/C^*. Moreover, G^*/B^* is of largest possible order among all finite groups G having an elementary abelian normal p-subgroup $A \leq \Phi(G)$ such that $G/A \cong H$; and G^*/C^* is of largest possible order among all finite n-generator groups G having an elementary abelian normal p-subgroup A such that $G/A \cong H$ and G splits over A. Gaschütz also shows ([3], Satz 8) that if the Sylow p-subgroups of H have order p^m and $|C^*| = p^s$ then $s \equiv 1 (\bmod\ p^m)$.

With this notation, we now show

Corollary 9. Let H be a finite p-soluble group, where p divides $|H|$, and let s be the largest positive integer for which there is a group G with an elementary abelian normal subgroup A of order p^s such that $A \leq \Phi(G)$ and $G/A \cong H$. Let P be a Sylow p-subgroup of H and say $|P| = p^m$. If P is abelian and the minimum number of generators of P is d then $s = 1 + rp^m$, where r is an integer such that $0 \leq r < d$. In particular, if P is cyclic then $s = 1$.

Proof. Let n be a positive integer such that H is an n-generator group and let

$$1 \to A^* \to G^* \to H \to 1$$

be the universal extension mentioned above. Then we may suppose that $G = G^*/B^*$, where B^* is as above, and $A = A^*/B^*$. Then $|A| = p^s$ and, by the result of Gaschütz mentioned above ([3], Satz 8), $s \equiv 1 \pmod{p^m}$. Let $\bar{P}$ be a Sylow p-subgroup of G. By hypothesis, $\bar{P}/A$ is abelian, so that $\bar{P}' \leq A$. Certainly G is p-soluble and so, by Corollary 7, $A \leq \Phi(\bar{P})$. Hence $\bar{P}/\Phi(\bar{P}) \cong P/\Phi(P)$, which has order p^d, by the Burnside Basis Theorem ([4], III.3.15, p.273). Therefore $\bar{P}$ is a d-generator group. Now it follows from Schreier's Theorem ([7], 8.4.13, p.203) that A is a $(1 + (d-1)p^m)$-generator group, hence that $s \leq 1 + (d-1)p^m$.

In view of the remarks above, we have as an immediate consequence

Corollary 10. Let H be a finite n-generator p-soluble group, where p divides $|H|$, and let P be a Sylow p-subgroup of H. Let t be the largest positive integer for which there is an n-generator semi-direct product of an elementary abelian group of order p^t by H. If P is abelian, $|P| = p^m$ and the minimum number of generators of P is d then $t = (n-1)|H| - rp^m$, where r is an integer such that $0 \leq r < d$. In particular, if P is cyclic then $t = (n-1)|H|$.

For instance, there is a 2-generator semi-direct product of an elementary abelian group of order 3^{24} by the symmetric group Σ_4 of degree 4.

Remark. Corollaries 7 and 9 both fail in the absence of the condition of p-solubility. To see this let H be as in Theorem 4 (so that by Proposition 5, H is not p-soluble), and let s, G and A be defined as in Corollary 9. Let P be a Sylow p-subgroup of G. Since A and G have the properties specified in (III), the proof of Theorem 4 applies to show that $A \not\leq \Phi(P)$. Since P/A is cyclic, $P' \leq A \leq P \cap \Phi(G)$; but the conclusion of Corollary 7 is violated. Again, if $s = 1$ then, as in the proof of Theorem 4, it follows from the fact that $|A| = p$ that $A \leq Z(G)$. But then $A \leq Z(P)$, and so by Theorem 3, $A \leq \Phi(P)$, in contradiction to the remark above. Hence $s > 1$, even though the Sylow p-subgroups of H are cyclic.

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A18

On finite insoluble groups with nilpotent maximal subgroups

By a celebrated result of J.G. Thompson ([17]; see also [5], p.445, Satz IV.7.4) a finite group is soluble if it has a nilpotent maximal subgroup of odd order. An elementary consequence ([10], Lemma 1 and Corollary) is that if a finite insoluble group G has a nilpotent maximal subgroup M, then $M = N_G(T)$ for some Sylow 2-subgroup T of G; hence every nilpotent maximal subgroup of G is conjugate in G to M.

Examples are known of insoluble groups which have nilpotent maximal subgroups: some of these examples will be considered later in this paper. In these the nilpotent maximal subgroups are actually the Sylow 2-subgroups, and we begin by showing that this phenomenon is in some sense typical.

Theorem 1. Suppose that G is a finite insoluble group with a nilpotent maximal subgroup M. Let T be the unique Sylow 2-subgroup of M and U the unique 2-complement of M. Then U is normal in G, $Z(U) \leq Z(G)$, $G/Z(U) \cong G/U \times U/Z(U)$ and G/U is an insoluble group whose Sylow 2-subgroups are maximal subgroups. In particular, if $Z(G) = 1$ then M is a Sylow 2-subgroup of G.

Proof. Suppose first that G is semi-simple (in the sense of H. Fitting): that is that G has no non-trivial soluble normal subgroup. Then every non-trivial Sylow subgroup of M has M as its normalizer in G. Moreover, M is a Hall subgroup of G; for if p were a common prime divisor of $|M|$ and $|G : M|$, the Sylow p-subgroup of M would be normal in G. By the theorem of Thompson mentioned above, $T \neq 1$. Suppose that $U \neq 1$; then M is not a Sylow subgroup of G. Hence, by a theorem of H. Wielandt ([20]; or [5], p.444, Satz IV.7.3), there is a normal subgroup K of G such that $G = MK$ and $M \cap K = 1$. Let q be a prime divisor of $|K|$ and Q a Sylow q-subgroup of K. By Frattini's Lemma, $G = N_G(Q)K$. Then $N_K(Q)$ is normal in $N_G(Q)$ and $N_G(Q)/N_K(Q) \cong G/K \cong M$. Thus $N_K(Q)$ is a normal Hall subgroup of $N_G(Q)$ and, by the Schur-Zassenhaus Theorem, $N_G(Q)$ has a subgroup $M^* \cong M$. By another result of Wielandt ([19];or [5], p.285, Satz III.5.8), M^* is conjugate in G to M, and therefore M^* is a maximal subgroup of G. But then $N_G(Q) \geq \langle M^*, Q\rangle = G$, so that Q is a non-trivial normal q-subgroup of G, which is contrary to the hypothesis that G is semi-simple. Therefore $U = 1$ and M is a Sylow 2-subgroup of G. There is nothing more to prove in this case.

Now consider the general case. Let L be the largest soluble normal subgroup of G. Since G is insoluble, $L \leq M$, and by what we have proved above, M/L is a Sylow 2-subgroup of G/L. Hence $U \leq L$. Since $U \leq L \leq M$, U is the unique 2-complement of L and therefore U is normal in G. Moreover, $M/U \cong T$ and so, because M/L is a Sylow 2-subgroup of G/L, M/U is a Sylow 2-subgroup of G/U. Now $M = T \times U \leq C_G(U)U$, which is normal in G. But M is certainly not normal in G, so it follows that $C_G(U)U = G$. Therefore $Z(U) \leq Z(G)$. Also

$$\begin{aligned} G/Z(U) &= C_G(U)/Z(U) \times U/Z(U) \\ &\cong G/U \times U/Z(U); \end{aligned}$$

and G/U is an insoluble group with a maximal subgroup M/U which is a Sylow 2-subgroup of G/U.

Finally, if $Z(G) = 1$ then $Z(U) = 1$ and therefore, since U is nilpotent, $U = 1$. Hence in this case M is a Sylow 2-subgroup of G.

Corollary 2. Suppose that G is a finite insoluble group with a nilpotent maximal subgroup M. Let $L = F(G)$, the Fitting subgroup of G.

Then G/L has a unique minimal normal subgroup K/L, K/L is non-abelian, and G/K is a 2-group.

Proof. Certainly $L \leq M$. Let K/L be a chief factor of G. Then K is not nilpotent, so that $MK = G$ and $G/K \cong M/M \cap K$. Hence K is insoluble, and so K/L is non-abelian. If G/L had another minimal normal subgroup, say K^*/L, then $K \cap K^* = L$ and G/K, G/K^* would both be nilpotent. But then G/L would be nilpotent, which is false. Thus K/L is the unique minimal normal subgroup of G/L. Certainly $Z(G/L) = 1$, so that, since G/L is an insoluble group with a nilpotent maximal subgroup M/L, Theorem 1 shows that M/L is a Sylow 2-subgroup of G/L. Therefore, since $L \leq M \cap K$, G/K is a 2-group.

We shall see in Lemma 7 that here G/K can be any finite 2-group. First we examine some known examples of insoluble groups with nilpotent maximal subgroups. We refer to the known list of all subgroups of the simple group $PSL(2,p^f)$, where p is a prime, f a positive integer and $p^f > 3$ (see [5], p.213, Satz II.8.27). Together with an elementary arithmetical result ([18], Lemma 3), this shows that the Sylow 2-subgroups are maximal subgroups of $PSL(2,p^f)$ if and only if $f = 1$ and p is a Fermat or Mersenne prime with $p \geq 17$. (The first observation that a non-abelian simple group can have a nilpotent maximal subgroup seems to be due to N. Itô [6].) In these cases, the Sylow 2-subgroups are dihedral groups ([5], p.196, Satz II.8.10).

In his paper [18], Thompson proved the following converse result:

(I). Let G be a finite insoluble group with a nilpotent maximal subgroup M. If the Sylow 2-subgroup of M is either dihedral or generalized quaternion then G has a subgroup G_o such that $|G : G_o| \leq 2$, $G_o > F(G)$ and $G_o/F(G) \cong PSL(2,q)$, where

(i) q is a Fermat or Mersenne prime > 7,

or (ii) $q = 9$,

or (iii) $q = 7$, in which case $|G : G_o| = 2$

To this one ought to add the remark that if $q = 9$ then also $|G : G_o| = 2$; for the Sylow 2-subgroups of $PSL(2,9)$ are dihedral of order 8 and are contained in subgroups of $PSL(2,9)$ isomorphic to the symmetric group Σ_4 of degree 4 ([5], p.202, Satz II.8.18). As a matter of fact, by an extension by Z. Janko [7] of a theorem of W.E. Deskins [3] and the result of Thompson mentioned at the beginning of this paper, the Sylow 2-subgroup T of M in (I) must have class ≥ 3 (see [5], p.445, Satz IV.7.4); and therefore $|T| \geq 16$.

It is not difficult to list the possibilities for $G/F(G)$ in (I). The following two results are easy and the proofs are omitted.

Lemma 3. Suppose that the group G has a normal subgroup H such that $Z(H) = 1$ and G/H is simple. If H is not a direct factor of G then G can be embedded in Aut H.

Lemma 4. Suppose that G_o is a subgroup of index 2 in the finite group G. If the Sylow 2-subgroups of G_o are maximal subgroups of G_o then the Sylow 2-subgroups of G are maximal subgroups of G.

An obvious consequence of Lemma 4 is

Corollary 5. Suppose that K is a normal subgroup of G such that G/K is a 2-group. If the Sylow 2-subgroups of K are maximal subgroups of K then the Sylow 2-subgroups of G are maximal subgroups of G.

In (I) the largest soluble normal subgroup of G must of course lie in M and must therefore coincide with $F(G)$. Thus, for the purpose of considering the possibilities for $G/F(G)$ in (I), we may as well assume that $F(G) = 1$. Then G is a finite semi-simple group in which, by Theorem 1, the Sylow 2-subgroups are maximal subgroups. By Lemma 3, G can be embedded in Aut G_o. It is known (see O. Schreier and B.L. van der Waerden [11], L-K. Hua [4], Appendix, R. Steinberg [16]) that when q is an odd prime, Aut $PSL(2,q) \cong PGL(2,q)$; and of course $PGL(2,q)$ contains a unique subgroup of index 2 which is isomorphic to

$PSL(2,q)$. Therefore in cases (i) and (iii) of (I), if $|G : G_o| = 2$, we must have $G \cong PGL(2,q)$. It is known that the Sylow 2-subgroups of $PGL(2,q)$ are dihedral whenever q is odd (see R.W. Carter and P. Fong [2]).

By Lemma 4 and the remarks above, the groups $PGL(2,q)$ actually occur with q as in (i). As Thompson pointed out, the group $PSL(2,7)$ does not occur as G in (I). However, the group $PGL(2,7)$ does occur, for the Sylow 2-subgroups of $PGL(2,7)$ are maximal subgroups. Suppose to the contrary that there were a proper subgroup R of $G = PGL(2,7)$ properly containing a Sylow 2-subgroup T of G. Let G_o be the subgroup of G isomorphic to $PSL(2,7)$. Then $T \cap G_o$ is a Sylow 2-subgroup of G_o and $R \cap G_o$ would be a proper subgroup of G_o properly containing $T \cap G_o$. This would imply that $R \cap G_o \cong \Sigma_4$ (see [5], p.213, Satz II.8.27) and hence, since Σ_4 is complete ([15] §15.3), that $R \cong \Sigma_4 \times W$ with $|W| = 2$. But this would contradict the fact that T is dihedral.

There remains the case (ii) of (I), with $q = 9$. This is rather more delicate. We have pointed out above that the group $PSL(2,9)$ does not occur. It is well known that $PSL(2,9) \cong A_6$, the alternating group of degree 6 ([5], p.183, Satz II.6.14). There is a natural identification of Aut A_6 and Aut Σ_6 (see [15] §11.4). By Lemma 3, we may identify Σ_6 with a subgroup of Aut A_6. Then $|\text{Aut } A_6/\Sigma_6| = 2$ ([15], loc.cit.). Thus the groups G which can occur in (I) with $F(G) = 1$ and $q = 9$. all appear as subgroups of index 2 in Aut A_6. The group Σ_6 is not one which occurs in (I), for the Sylow 2-subgroupsof Σ_6 are obviously contained in subgroups isomorphic to $\Sigma_4 \times W$, where $|W| = 2$. However, Aut A_6 has other subgroups of index 2, for by Lemma 3, Aut A_6 has a subgroup isomorphic to $PGL(2,9)$; and since the Sylow 2-subgroups of $PGL(2,9)$ are dihedral, $PGL(2,9) \not\cong \Sigma_6$. Thus Aut A_6/A_6 is a four-group, and so there remain two subgroups of Aut A_6 to consider, one isomorphic to PGL(2,9) and another. In fact, the Sylow 2-subgroups of PGL(2,9) are maximal subgroups: we can see this by the same argument as for PGL(2,7).

In [21], W.J. Wong defines, for each integer q of the form $q = p^{2f}$, where p is an odd prime and f a positive integer, a group H(q) containing a subgroup of index 2 isomorphic to PSL(2,q). The Sylow 2-subgroups of H(q) are semidihedral (also called quasidihedral). By Lemma 3, there is a group G of index 2 in Aut A_6 isomorphic to H(9); and since the Sylow 2-subgroups of H(9) are isomorphic neither to the Sylow 2-subgroups of Σ_6 nor to the Sylow 2-subgroups of PGL(2,9), this group G must be the one remaining subgroup of index 2 in Aut A_6. Since the Sylow 2-subgroups of G are neither dihedral nor generalized quaternion, G does not occur in (I). Thus we now have a complete list of the possible groups G/F(G) in (I).

We note that the Sylow 2-subgroups of H(9) are maximal subgroups; we can use the same argument as for PGL(2,7) and PGL(2,9). Therefore it is natural to ask how (I) is affected by allowing the Sylow 2-subgroup of M to be semidihedral. This question was settled by J. Randolph [9], who showed

(II). The conclusion of (I) is unchanged if the Sylow 2-subgroup of M is allowed to be semidihedral.

Then the remarks above show that the only additional group which occurs as G/F(G) in (I) and (II) is H(9): this was noted by Randolph. If now we take account of results of A.R. Camina and T.M. Gagen [1] and Wong [21] (Theorem 1; or [5], p.424, Satz IV.3.5), together with [5], p.90, Satz I.14.9 and the Feit-Thompson Theorem, the conclusions can be summarized in

Theorem 6. (Reformulation of the results of Thompson [17] and Randolph [9].) Let G be a finite insoluble group with a nilpotent

maximal subgroup M. If the Sylow 2-subgroup of M is metacyclic then $G/F(G)$ is isomorphic to one of the following groups :
$PSL(2,q)$, $PGL(2,q)$, $PGL(2,9)$, $H(9)$, $PGL(2,7)$,
where q denotes a Fermat or Mersenne prime ≥ 17. In all these groups the Sylow 2-subgroups are maximal subgroups; in all but $H(9)$ they are dihedral, while in $H(9)$ they are semidihedral.

Thompson remarked that it appeared likely that the conclusion of (I) would remain valid with no assumptions whatsoever on the structure of the Sylow 2-subgroup of M. In a private communication to the author, J.L. Alperin indicated that this could not be true. The remarks above, together with Lemma 4, show that Aut A_6 provides a counter-example. In fact, as Alperin suggested, one has

Lemma 7. Let G be a finite insoluble group in which the Sylow 2-subgroups are maximal subgroups. Let X be any finite 2-group and let $W = G \wr X$, the regular wreath product of G by X. Then the Sylow 2-subgroups of W are maximal subgroups.

Proof. Let K be the base group of W, so that

$$K = \mathrm{Dr} \prod_{x \in X} G_x ,$$

where, for each $x \in X$, G_x is a copy of G; $W = XK$, $X \cap K = 1$ and for all $x,y \in X$,

$$G_x^y = G_{xy} .$$

Let T be a Sylow 2-subgroup of W containing X.

Then $$T = X(T \cap K),$$

and $T \cap K$ is a Sylow 2-subgroup of K. For each $x \in X$,

$T \cap G_x$ is a Sylow 2-subgroup of G_x and

$$T \cap K = \mathrm{Dr} \prod_{x \in X} (T \cap G_x) .$$

Suppose that $T \leq V < W$. Then

$$V = X(V \cap K)$$

and, for each $x \in X$,

$$T \cap G_x \leq V \cap G_x \leq G_x ,$$

so that by hypothesis

$$V \cap G_x = \text{either } T \cap G_x \text{ or } G_x .$$

If for some $y \in X$, $V \cap G_y = G_y$, then $G_y \leq V$ and therefore, since $X \leq V$, $G_x \leq V$ for all $x \in X$, Hence $XK = W \leq V$, contrary to hypothesis. Thus

$$V \cap G_x = T \cap G_x \text{ for all } x \in X.$$

Let the projection of $V \cap K$ on G_x be U_x. Then $V \cap G_x$ is normal in U_x. But $T \cap G_x$ is a non-normal maximal subgroup of G_x and therefore

$$U_x = T \cap G_x \text{ for all } x \in X.$$

Hence $V \cap K \leq \mathrm{Dr}\prod_{x \in X} U_x = \mathrm{Dr}\prod_{x \in X} (T \cap G_x) = T \cap K.$

It follows that $V \cap K = T \cap K$ and so $V = T$. Thus T is a maximal subgroup of W.

If now in Lemma 7 we choose for G one of the simple groups $PSL(2,q)$ listed in Theorem 6, then W is semi-simple, the base group K of W is the unique minimal normal subgroup of W, and of course $W/K \cong X$. This shows that in Corollary 2 the 2-residual of the group in question can be any finite 2-group. The general problem of classification of semi-simple groups whose Sylow 2-subgroups are maximal subgroups seems at present to be quite intractable.

We end this paper by showing that under suitable circumstances in Theorem 1, U is itself a direct factor of G (Theorem 12). However, this is not true in general (Corollary 11). We need some preliminary information about multiplicators.

Let G be a finite group. We shall call a finite group $\hat{G}$ a cover of G (in German, Darstellungsgruppe von G) if $\hat{G}$ has a subgroup $A \leq \hat{G}' \cap Z(G)$ such that $\hat{G}/A \cong G$ and $|\hat{G}|$ is as large as possible subject to these conditions. Then $A \cong H^2(G,\mathbb{C}^\times)$, the 2nd cohomology group of G with coefficients in the trivial G-module $\mathbb{C}^\times$ (the multiplicative group of the field $\mathbb{C}$ of complex numbers). The group $H^2(G,\mathbb{C}^\times)$ is called the multiplicator of G: we shall denote it by M(G). (See I. Schur [12], and [5], § V.23.) We include proofs of the following two lemmas in the spirit of [5].

Lemma 8 (Schur [13], p.108). Let T be a semidihedral group of order 2^{n+1}, where n is an integer ≥ 3. Then T has trivial multipicator.

Proof. The group T can be defined by a set of 2 generators with 3 relations ([5], p.90, Satz I.14.9b)(4)). Hence M(T) is cyclic ([5], p.642, Satz V.25.2a)). Since T has a cyclic subgroup of index 2, it follows that $|M(T)| \leq 2$ ([5], p.635, Satz V.23.9b)).

Suppose that $|M(T)| = 2$ and let $\hat{T}$ be a cover of T. Then $\hat{T}$ has a subgroup $A \leq \hat{T}' \cap Z(\hat{T})$ such that $\hat{T}/A \cong T$ and $|A| = 2$. Let $\bar{T} = \hat{T}/A$ and let the usual bar convention apply. Then there are in $\hat{T}$ elements x,y such that $\bar{T} = \langle \bar{x},\bar{y} \rangle$, where

$$\bar{x}^{2^n} = 1 = \bar{y}^2 \quad \text{and} \quad \bar{x}^{\bar{y}} = \bar{x}^{2^{n-1}-1}.$$

Then $\hat{T} = \langle x,y \rangle$ ([5], p.272, Satz III.3.12); and if $A = \langle a \rangle$ then

$$x^{2^n} = a^r, \quad y^2 = a^s \quad \text{and} \quad x^y = a^t x^{2^{n-1}-1},$$

where each of r,s,t is either 0 or 1. Now

$$x = x^{y^2} = a^t(a^t x^{2^{n-1}-1})^{2^{n-1}-1} = a^{2^{n-1}t} x^{2^{2n-2}-2^n+1}.$$

Since $a^2 = 1$, $x^{2^{n+1}} = 1$, and since $n \geq 3$, $2n-2 \geq n+1$. Thus the equations above imply that

$$x^{2^n} = 1.$$

Since $\bar{x}$ has order 2^n, it follows that $a \notin \langle x \rangle$.

Now

$$[x,y] = a^t x^{2^{n-1}-2},$$

and since $a \in Z(\hat{T})$,

$[x,y]^x = [x,y]$ and $[x,y]^y = a^t(a^t x^{2^{n-1}-1})^{2^{n-1}-2} = [x,y]^{2^{n-1}-1}$.

Hence $\langle [x,y] \rangle$ is normal in $\hat{T}$ and therefore

$$\hat{T}' = \langle [x,y] \rangle = \langle a^t x^{2^{n-1}-2} \rangle .$$

Since $a \in \hat{T}'$ is follows that

$$a = (a^t x^{2^{n-1}-2})^m \quad \text{for some integer } m.$$

Since $a \notin \langle x \rangle$, $t = 1$, m is odd and

$$x^{(2^{n-1}-2)m} = 1.$$

But since $2^{n-1}-2$ is twice an odd integer, this contradicts the fact that x has order 2^n with $n \geq 3$. Therefore $|M(T)| = 1$.

Lemma 9 (Schur [13], pp.121-2). The multiplicator of PGL(2,9) has order 2.

Proof. Let $G = PGL(2,9)$: then $|G| = 2^4 \times 3^2 \times 5$. Hence ([5], p.642, Satz V.25.1 and p.643, Satz V.25.3a)) the only possible prime divisors of $|M(G)|$ are 2 and 3. Let $K = A_6$ and, as before, identify Σ_6, G and $H(9)$ with subgroups of index 2 in Aut K.

Let $u = (123)$, $v = (456)$ and $U = \langle u \rangle \times \langle v \rangle$: then $U < K$ and U is a Sylow 3-subgroup of G. It is easy to see that $N_K(U) = \langle (1425)(36) \rangle U$, and clearly $|N_G(U) : N_K(U)| = 2$. Now

$$\text{Aut } K = \langle \alpha \rangle \Sigma_6 ,$$

where

$$(12)^\alpha = (12)(34)(56), \quad (23)^\alpha = (23)(45)(16), \quad (34)^\alpha = (12)(35)(46),$$

$$(45)^\alpha = (23)(56)(14) \text{ and } (56)^\alpha = (12)(45)(36);$$

and α has order 2 (P.J. Lorimer [8]). Since the semidihedral group of order 16 does not split over its (unique) dihedral subgroup of order 8, $\alpha \notin H(9)$. Therefore

$$\langle \alpha \rangle K = G.$$

Now it is easy to verify that $\alpha(253) \in N_K(U)$ and $\alpha(253)$ has order 8. Thus

$$N_K(U) = \langle k \rangle U, \quad \text{where } k = \alpha(253).$$

The action of $\langle k \rangle$ on U by conjugation determines a representation (with respect to the base u,v of U)

$$\Theta : \langle k \rangle \rightarrow GL(2,3).$$

Let $\Theta^* = \Theta \otimes \Theta : \langle k \rangle \to GL(4,3)$.

Then one verifies (using the base $u\otimes u$, $u\otimes v$, $v\otimes u$, $v\otimes v$ of $U\otimes U$) that

$$k\Theta^* = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{pmatrix} .$$

Since 1 is not an eigenvalue of $k\Theta^*$, it follows ([5], p.644, Hilfssatz V.25.4) that 3 does not divide $|M(G)|$.

Therefore, since the Sylow 2-subgroups of G are dihedral, $|M(G)| \leq 2$ ([5], p.642, Satz V.25.1 and p.646, Satz V.25.6). In fact $|M(G)| = 2$ because G does have a cover of order $2|G|$. To see this note that $|GF(3^4)^\times| = 80$ and let λ be an element of order 16 in $GF(3^4)^\times$. Let

$$x = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \in SL(2,3^4).$$

Then $x^2 \in SL(2,9)$ and one verifies that the subgroup $\langle x \rangle SL(2,9)$ of $SL(2,3^4)$ is a cover of G.

Lemma 10. The multiplicator of H(9) has order 3.

Proof. Since $|H(9)| = 2^4 \times 3^2 \times 5$ and the Sylow 2-subgroups of H(9) are semidihedral and the Sylow 3-subgroups of H(9) are elementary abelian, it follows (by Lemma 8 and [5], p.642, Satz V.25.1, p.643, Satz V.25.3a) and p.652, Satz V.25.11) that 3 is the only possible prime divisor of $|M(H(9))|$ and that $|M(H(9))| \leq 3$. We shall show that $|M(H(9))| = 3$ by constructing explicitly a cover of H(9).

It is known that $|M(A_6)| = 6$ ([5], p.646, Satz V.25.7); and Schur ([14], p.242) has written down explicit generators and defining relations for the (unique) cover $\hat{A}_6$ of A_6:

$$\hat{A}_6 = \langle x_1, x_2, x_3, x_4 \rangle, \text{ with relations}$$

$$x_1^{\,3} = x_2^{\,2} = x_3^{\,2} = x_4^{\,2} = (x_1x_2)^3 = (x_1x_3)^2 = (x_2x_3)^3 = (x_3x_4)^3 = w^3,$$

$$(x_1x_4)^2 = w, \quad x_2x_4 = w^3x_4x_2, \quad wx_i = x_iw \text{ for all } i, \quad w^6 = 1.$$

Let $K = \hat{A}_6/\langle w^3\rangle$: then K has a subgroup $L = \langle w\rangle/\langle w^3\rangle \leq K' \cap Z(K)$ with $|L| = 3$ and $K/L \cong A_6$. Now K has a presentation of the form

$K = \langle y_1, y_2, y_3, y_4\rangle$, with defining relations

$$y_1^{\,3} = y_2^{\,2} = y_3^{\,2} = y_4^{\,2} = (y_1y_2)^3 = (y_1y_3)^2 = (y_2y_3)^3 = (y_3y_4)^3 = 1,$$

$$(y_1y_4)^2 = z, \quad y_2y_4 = y_4y_2, \quad zy_i = y_iz \text{ for all } i, \quad z^3 = 1.$$

Here $L = \langle z\rangle$. Let $\bar{K} = K/\langle z\rangle$ and let the bar convention apply. We identify $\bar{K}$ with A_6 by identifying

$\bar{y}_1 = (123)$, $\bar{y}_2 = (12)(34)$, $\bar{y}_3 = (12)(45)$, $\bar{y}_4 = (12)(56)$:
see [5],p.138, Beispiel I.19.8.

We begin by observing that the Sylow 3-subgroups of K (which have order 3^3) are non-abelian of exponent 3. Let U be the Sylow 3-subgroup of K such that

$$\bar{U} = \langle (123)\rangle \times \langle (456)\rangle$$

(which is a Sylow 3-subgroup of $\bar{K}$). Then

$$\bar{U} = \langle \bar{y}_1\rangle \times \langle \bar{y}_3\bar{y}_4\rangle .$$

Suppose that $[y_1, y_3y_4] = 1$. Then

$$(y_1y_3y_4)^2 = y_1^{\,2}(y_3y_4)^2$$

$$= y_1^{-1}y_4^{-1}y_3^{-1} \quad \text{, since } y_1^{\,3} = 1 = (y_3y_4)^3.$$

Since $\quad y_1y_4 = z(y_1y_4)^{-1}$,

conjugation by $\quad y_4 = y_4^{-1}$ gives

$$y_4y_1 = z(y_4y_1)^{-1}, \text{ and so}$$

$$(y_1y_3y_4)^2 = z^{-1}y_4y_1y_3^{-1} . \tag{i}$$

But also on the supposition above,

$$(y_1y_3y_4)^3 = 1, \quad \text{so that}$$

$$\begin{aligned}(y_1y_3y_4)^2 &= (y_1y_3y_4)^{-1}\\ &= y_4^{-1}\,y_3^{-1}\,y_1^{-1}\,. \end{aligned} \qquad \text{(ii)}$$

Since $y_3{}^2 = 1 = y_4{}^2$, it follows from (i) and (ii) that

$$(y_1y_3)^2 = z,$$

which contradicts the relation $(y_1y_3)^2 = 1$. Hence U is non-abelian and $U = \langle y_1, y_3y_4\rangle$. Since $|U| = 3^3$, U is regular ([5], p.322, Satz III.10.2) and therefore, since y_1 and y_3y_4 have order 3, U has exponent 3 ([5], p.324, Hauptsatz III.10.5).

Let $J = \langle y_1, y_2, y_3\rangle < K$.

Then J is mapped by the natural homomorphism $\nu : K \to \bar{K}$ onto

$$\bar{J} = \langle \bar{y}_1, \bar{y}_2, \bar{y}_3\rangle = A_5.$$

Since $\qquad y_1{}^3 = y_2{}^2 = y_3{}^2 = (y_1y_2)^3 = (y_1y_3)^2 = (y_2y_3)^3 = 1,$

J is itself a homomorphic image of A_5 ([5], p.138, Beispiel I.19.8). Therefore ν maps J isomorphically onto $\bar{J}$.

Now Aut $\bar{K} = \langle\alpha\rangle\,\Sigma_6$, where α is defined as in the proof of Lemma 9. As before, we identify Σ_6, PGL(2,9) and H(9) in the natural way with the 3 subgroups of index 2 in Aut $\bar{K}$. Let

$$\bar{\gamma} = \alpha(12) \in \text{Aut } \bar{K}.$$

Then $\qquad \bar{\gamma}^2 = (34)(56) \in \bar{K}.$

The group $\langle\bar{\gamma}\rangle\bar{K}$ is a subgroup of index 2 in Aut $\bar{K}$ and is distinct from both Σ_6 and $\langle\alpha\rangle\bar{K} = \text{PGL}(2,9)$. Hence

$$\langle\bar{\gamma}\rangle\bar{K} = H(9).$$

We shall show that $\bar{\gamma}$ is induced by an automorphism γ of K: then

$$\overline{y^\gamma} = \bar{y}^{\bar{\gamma}} \quad \text{for all } y \in K.$$

One verifies that

$$\begin{cases} \bar{y}_1^{\bar{\gamma}} = (253)(146) = (\bar{y}_2\bar{y}_4)^{\bar{y}_3\bar{y}_1^{-1}} \bar{y}_2^{\bar{y}_1} \\ \bar{y}_2^{\bar{\gamma}} = (36)(45) = (\bar{y}_2\bar{y}_4)^{\bar{y}_3\bar{y}_2}, \\ \bar{y}_3^{\bar{\gamma}} = (23)(14) = \bar{y}_2^{\bar{y}_1}, \text{ and} \\ \bar{y}_4^{\bar{\gamma}} = (35)(46) = (\bar{y}_2\bar{y}_4)^{\bar{y}_3}. \end{cases}$$

Let

$$\begin{cases} y_1' = (y_2y_4)^{y_3y_1^{-1}} y_2^{y_1}, \\ y_2' = (y_2y_4)^{y_3y_2}, \\ y_3' = y_2^{y_1}, \text{ and} \\ y_4' = (y_2y_4)^{y_3}. \end{cases}$$

Then it is enough to show that there is an automorphism γ of K such that

$$y_i^{\gamma} = y_i' \quad \text{for all } i;$$

and this will be so if $\langle y_1', y_2', y_3', y_4' \rangle = K$ and y_1', y_2', y_3', y_4' satisfy the same relations as y_1, y_2, y_3, y_4 (in order).

Since

$$\langle \overline{y_1'}, \overline{y_2'}, \overline{y_3'}, \overline{y_4'} \rangle = \langle \bar{y}_1^{\bar{\gamma}}, \bar{y}_2^{\bar{\gamma}}, \bar{y}_3^{\bar{\gamma}}, \bar{y}_4^{\bar{\gamma}} \rangle = \bar{K}$$

and z is a non-generator of K ([5], p.272, Satz III.3.12) it is certainly true that

$$\langle y_1', y_2', y_3', y_4' \rangle = K.$$

Since $\overline{y_1'} = \bar{y}_1^{\bar{\gamma}}$ has order 3, y_1' is a 3-element of K and therefore, since the Sylow 3-subgroups of K have exponent 3,

$$y_1'^3 = 1.$$

Also, since y_2 and y_2y_4 have order 2,

$$y_2'^2 = y_3'^2 = y_4'^2 = 1.$$

The same argument as for y_1' shows that

$$(y_i'y_{i+1}')^3 = 1 \quad \text{for } i = 1,2,3.$$

Next, since y_2 has order 2,

$$y_1'y_3' = (y_2y_4)^{y_3y_1^{-1}},$$

and therefore, since y_2y_4 has order 2,

$$(y_1'y_3')^2 = 1.$$

Now let

$$w = (y_2y_4)^{y_3}.$$

Then

$$y_1'y_4' = w^{y_1^{-1}} y_2^{y_1} w.$$

Let

$$v = (y_1'y_4')^{w^{y_1^{-1}}} = y_2^{y_1} w w^{y_1^{-1}}.$$

Since $\quad y_3^{\,2} = (y_1y_3)^2 = 1,$

$$y_3y_1^{-1} = (y_1y_3)^{-1} = y_1y_3.$$

Also $y_4^2 = 1$ and $y_4y_1 = z(y_4y_1)^{-1}$, so that

$$y_4^{y_1} = (y_4y_1)^{-1}y_1 = z^{-1}y_4y_1^2.$$

Therefore

$$\begin{aligned} w w^{y_1^{-1}} &= (y_2y_4(y_2y_4)^{y_1})^{y_3} \\ &= (y_2y_4(y_4y_2)^{y_1})^{y_3} \\ &= (y_2y_4z^{-1}y_4y_1^2y_2^{\,y_1})^{y_3} \\ &= z^{-1}(y_2y_1y_2y_1)^{y_3}. \end{aligned}$$

Since $(y_1y_2)^3 = 1$, $(y_2y_1)^3 = 1$ and so

$$ww^{y_1^{-1}} = z^{-1}((y_2y_1)^{-1})^{y_3}$$

$$= z^{-1}(y_1^{-1}\ y_2)^{y_3} .$$

Hence

$$v = z^{-1}y_2^{\ y_1}(y_1^{-1}y_2)^{y_3} .$$

Now

$$zv \in J,$$

and one verifies that

$$\overline{zv} = \bar{v} = (14)(35),$$

of order 2. Since ν maps J isomorphically onto $\bar{J}$, it follows that zv has order 2. Hence

$$v^2 = z^{-2}(zv)^2 = z.$$

Now (since w has order 2),

$$(y_1'y_4')^2 = (v^2)^{w^{y_1^{-1}}} = z.$$

Since $\overline{y_2'y_4'} = (\bar{y}_2\bar{y}_4)^{\bar{\gamma}}$, which has order 2, $y_2'y_4'$ has order either 2 or 6; and if the latter alternative holds then $z \in \langle y_2', y_4' \rangle$. But since y_2' and y_4' have order 2, it follows that if $y_2'y_4'$ has order 6 then $\langle y_2', y_4' \rangle$ is dihedral of order 12 and hence does not have a central element of order 3. Thus $y_2'y_4'$ has order 2 and

$$y_2'y_4' = y_4'y_2' .$$

It is certainly true that

$$zy_i' = y_i'z \text{ for all } i, \text{ and } z^3 = 1.$$

This completes the proof that there is an automorphism γ of K such that

$$y_i^{\gamma} = y_i' \text{ for all } i.$$

Since $\bar{\gamma}^2 = (34)(56) = \bar{y}_2\bar{y}_4$,

$$y_i^{\gamma^2} = z_i y_i^{y_2y_4} \quad \text{for all } i,$$

where each $z_i \in \langle z\rangle$. Since y_i has order 2 for $i = 2,3,4$ $z_i = 1$ and

$$y_i^{\gamma^2} = y_i^{y_2y_4} \quad \text{for } i = 2,3,4.$$

Now $(y_1y_3)^{\gamma^2} = z_1(y_1y_3)^{y_2y_4}$,

and since y_1y_3 has order 2, it follows that $z_1 = 1$ and so

$$y_1^{\gamma^2} = y_1^{y_2y_4} .$$

Thus γ^2 is conjugation in K by y_2y_4.

Now let G be the natural semidirect product of K by $\langle\gamma\rangle$. Since

$$y^{\gamma^2} = y^{y_2y_4} \quad \text{for all } y \in K$$

and y_2y_4 has order 2, the element $\gamma^2y_2y_4$ of G centralizes K.

Moreover, since

$$(\bar{y}_2\bar{y}_4)^{\bar{\gamma}} = (\bar{\gamma}^2)^{\bar{\gamma}} = \bar{y}_2\bar{y}_4$$

and y_2y_4 has order 2,

$$(y_2y_4)^{\gamma} = y_2y_4.$$

Hence

$$\gamma^2y_2y_4 \in Z(G).$$

Let

$$Y = \langle\gamma^2y_2y_4\rangle, \text{ a central subgroup of G of order 2.}$$

Since $z = (y_1y_4)^2$,

$$z^{\gamma} = (y_1'y_4')^2 = z , \quad \text{and so}$$

$L = \langle z\rangle$ is a central subgroup of G of order 3.

Clearly $G' = K$, and therefore

$$LY/Y \leq (G/Y)' \cap Z(G/Y),$$

with $|LY/Y| = 3$. Moreover, since γ induces $\bar{\gamma}$, it is easy to see that

$$G/LY \cong H(9).$$

Hence $|M(H(9))| = 3$ and G/Y is a cover of $H(9)$.

Remark. Since $|H(9)/H(9)'| = 2$ and $|M(H(9))| = 3$, $H(9)$ has a unique cover (Schur [13], pp.95-6).

Corollary 11. Let $G = \widehat{H(9)}$, the unique cover of $H(9)$. Then G has a nilpotent maximal subgroup M, and the unique 2-complement U of M is not a direct factor of G.

Proof. Let $A \leq G' \cap Z(G)$ with $G/A \cong H(9)$. By Lemma 10, $|A| = 3$. We know that the Sylow 2-subgroups of G/A are maximal subgroups. Let M/A be a Sylow 2-subgroup of G/A. Then M is a maximal subgroup of G and, since $A \leq Z(G)$, M is nilpotent. Moreover, A is the unique 2-complement U of M. Finally, since $1 \neq A \leq G'$, A is not a direct factor of G.

We note that by Theorem 6, if G is a finite semi-simple group with a Sylow 2-subgroup T which is matacyclic and is a maximal subgroup of G, then T is dihedral unless $G \cong H(9)$. This accounts for the hypothesis in the final result.

Theorem 12. Let G be a finite insoluble group with a nilpotent maximal subgroup M, T the unique Sylow 2-subgroup of M and U the unique 2-complement of M. Let $L = F(G)$: then, by Theorem 1, $U \leq L < M$. If M/L is dihedral then U is a direct factor of G.

Proof. Suppose the result false and let G provide a counter-example of least possible order. Since $U \leq L < M = T \times U$,

$$L = (T \cap L) \times U$$

and $T \cap L$ is normal in G. Let $\bar{G} = G/T \cap L$ and let the usual bar convention apply. Then $\bar{G}$ is an insoluble group with a nilpotent maximal subgroup $\bar{M}$. The 2-complement of $\bar{M}$ is $\bar{U}$. Since G/L is semi-simple, $F(\bar{G}) = \bar{L}$, and of course $\bar{M}/\bar{L}$ is dihedral. Hence if $T \cap L \neq 1$, the minimality of G implies that $\bar{U}$ is a direct factor of $\bar{G}$, say

$$\bar{G} = \bar{U} \times \bar{V}, \quad \text{where } T \cap L \le V \le G.$$

But then $G = UV$, U and V are normal in G, and $U \cap V = U \cap T \cap L = 1$, so that $G = U \times V$. This is contrary to hypothesis. Therefore $T \cap L = 1$, $L = U$ and T is dihedral.

We know that $G = C_G(U)U$ (see the proof of Theorem 1). Since $U \le M$, it follows that $M = C_M(U)U$. Thus $M/Z(U) = C_M(U)/Z(U) \times U/Z(U)$ and this is a maximal subgroup of $G/Z(U) = C_G(U)/Z(U) \times U/Z(U)$. Therefore $C_M(U)$ is a maximal subgroup of $C_G(U)$. Certainly $C_G(U)$ is insoluble and $C_M(U)$ is nilpotent. Moreover

$$C_M(U) = T \times Z(U).$$

Since now $U = F(G)$ and since $F(C_G(U)) \le F(G)$, because $C_G(U)$ is normal in G,

$$F(C_G(U)) = Z(U).$$

Hence if $C_G(U) < G$, the minimality of G implies that $Z(U)$ is a direct factor of $C_G(U)$, say

$$C_G(U) = H \times Z(U).$$

But then $G = HU$ and $H \cap U = H \cap C_G(U) \cap U = 1$; hence since $H \le C_G(U)$, $G = H \times U$. This is contrary to hypothesis. Therefore $U \le Z(G)$, and so in fact $U = Z(G)$.

Certainly $U \neq 1$. Let p be a prime divisor of $|U|$ and let P be a subgroup of U of order p. Now let $\bar{G} = G/P$ and let the bar convention apply. Then $\bar{G}$ is an insoluble group with a nilpotent maximal subgroup $\bar{M} = \bar{T} \times \bar{U}$; since $P \le Z(G)$, $F(\bar{G}) = \bar{U}$; and $\bar{T} \cong T$. Therefore the minimality of G implies that $\bar{U}$ is a direct factor of $\bar{G}$, say

$$\bar{G} = \bar{U} \times \bar{W}, \quad \text{where } P \leq W \leq G.$$

Then $G = UW$, with U and W normal in G, and $U \cap W = P$. Since $U \leq M$, $M = U(M \cap W)$. Moreover, $\bar{M} = \bar{U} \times \overline{M \cap W}$ is a maximal subgroup of $\bar{G} = \bar{U} \times \bar{W}$, and therefore $M \cap W$ is a maximal subgroup of W. Now W is an insoluble group with a nilpotent maximal subgroup $M \cap W$. Since $|U|$ is odd and W is normal in G, $T \leq W$. Hence $M \cap W = T \times P$, and clearly $F(W) = P$. Thus if $W < G$, the minimality of G implies that P is a direct factor of W, say

$$W = P \times Q.$$

But then $G = UW = UQ$ and $U \cap Q = U \cap W \cap Q = P \cap Q = 1$; hence, since $U = Z(G)$, $G = U \times Q$. This is contrary to hypothesis. Therefore $W = G$ and $|U| = p$.

If $U \not\leq G'$ then $U \cap G' = 1$ and p divides $|G/G'|$. Then G has a normal subgroup J such that $|G/J| = p$. Since $U = F(G)$, Corollary 2 shows that $U \not\leq J$. But then $G = J \times U$, which is contrary to hypothesis. Therefore $U \leq G'$ and so $U = G' \cap Z(G)$. It follows (Schur [12], p.31; [5], p.629, Hilfssatz V. 23.3) that p divides the order of the multiplicator of the semi-simple group G/U. Hence the Sylow p-subgroups of G/U cannot be cyclic ([5], p.642, Satz V.25.1 and p.643, Satz V. 25.3a)).

By Theorem 6 (applied to the group G/U), G/U is isomorphic to one of the groups $PSL(2,q)$, $PGL(2,q)$, $PGL(2,9)$, $PGL(2,7)$, where q denotes a Fermat or Mersenne prime ≥ 17. Then, since p is odd, the fact that the Sylow p-subgroups of G/U are not cyclic implies that $p=3$ and $G/U \cong PGL(2,9)$ ([5], p.191, Satz II.8.2 and p.196, Satz II.8.10). But by Lemma 9, the multiplicator of $PGL(2,9)$ is not divisable by 3. This final contradiction establishes the theorem.

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